Two interacting atoms in a quantum optical potential

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Katharina Renz
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bei A. Univ.-Prof. Dr. Helmut Ritsch

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Diese Diplomarbeit widme ich meiner Oma!

Contents

Chapter 1

Introduction

During the last decades a great deal of theoretical and experimental research was focused on cavity QED - the study of a single two-level atom interacting with a single or a few field modes of an optical resonator. Various configurations to analyze the atom-field dynamics have been investigated theoretically (cavity QED) [1, 2, 3]. Experiments combining cavity QED with optical trapping and cooling experiments have been realized in recent works [4, 5]. The basic idea for the research on light and matter is rooted in the Jaynes-Cummings model [6]. This at first only theoretical model of a two-level atom interacting with a single mode of the electromagnetic field was developed by E. T. Jaynes and F. W. Cummings in the $1960'ies$ in order to compare the predictions of Quantum Mechanics with those of the classical Maxwell equations. It could be realized experimentally with good adjustment to the theoretical idealization in the meanwhile. The idealization in this model is the restriction to a two-level system and to one single mode of the electromagnetic field. Although each atom has an infinite number of states with an infinite number of permitted transitions between them, it is possible to neglect all energy levels except the two levels whose energy difference is near the energy of the field. Such a two-level system can be described by the Pauli spinoperator σ_z and the transition frequency of the 2-level system ω_a. The operator-representation of the field is more complex. The simplest case that though describes basic aspects of the interaction is to disregard all field modes except for one. A single mode is comparatively easy to describe by the creation $a^\dagger$ and annihilation a operators. The Hamiltonian of the field contains the laser frequency ω_L and the operator product $a^\dagger a$ which counts the number of photons.

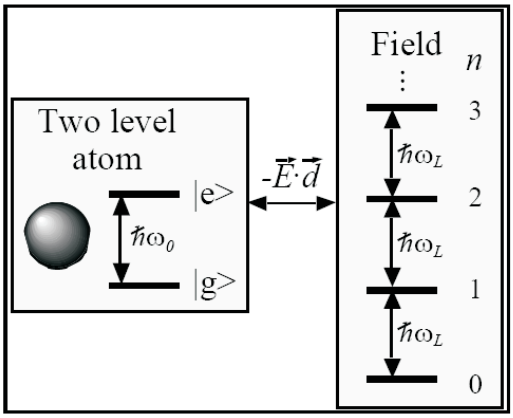

Figure 1.1: *The dipole-interaction between the 2-level atom and the electromagnetic field.*

The Jaynes-Cummings description of the interaction of the atom and the field mode is related to the classical description of the interaction between an atom and an electromagnetic field, namely the scalarproduct of the electric field vector $\vec{E}$ and the vector for the atomic dipole moment $\vec{d}$. According to the correspondence principle the classical variables are replaced by the corresponding quantum mechanical operators. The dipole operator has non-disappearing elements only between states with different parity and is therefore proportional to the x-component of the Pauli-matrices $\sigma_x = (\sigma^+ + \sigma^-)$. The field operator for a single mode is proportional to the sum of a creation and an annihilation operator $a+a^\dagger$. The resulting interaction operator $H_{JC} = -i\hbar g(x)\left(\sigma^+ a - a^\dagger \sigma^-\right)$, where only energy-conserving processes are taken into account, describes the processes of annihilation of a photon and simultaneous increase of the atomic energy and the reverse process of decreasing the atom's energy and simultaneous creation of a photon. Thus H_{JC} describes absorption and emission of photons by the atom. The interaction operator includes the coupling $g(x)$ of the atom to the field, which depends on the atomic dipole moment, the mode volume and the position of the atom in the mode. The energy difference between two states of the combined atom-field system is zero if the laser frequency ω_L equals the resonance frequency ω_a of the atomic transition. Otherwise the energy difference is just the detuning from resonance. The system contains an infinite series of such states which are called 'dressed states'.

The exchange of photon momentum between light and matter is used to control atomic motion. An atom interacting with an electromagnetic field will experience forces occurring on the one hand from the radiation pressure of the electromagnetic field and on the other hand from the electric dipole interaction.

The radiation pressure force arises from absorption and spontaneous emission of photons and can be described as the product of the transferred photon momentum $\hbar k$ and the photon scattering rate Γ of the atom, which is the rate of absorption-spontaneous emission cycles. This force can accelerate or slow down atoms respectively. The mechanism is based on the Doppler effect and therefore is called Doppler-cooling [7, 8]. The moving particle tends to absorb photons from the laser wave counterpropagating its velocity rather than from the copropagating wave, if the laser is tuned below resonance. The Doppler-cooling though is limited by the atomic linewidth such that after some cycles the force decreases and the minimum attainable temperature is reached. This minimum temperature is called the Doppler limit $k_B T \approx \hbar\Gamma$ [9, 10, 11]. The cooling force can be increased by imposing a magnetic field gradient on the system which varies the energy of the atomic states via the Zeeman effect. Nevertheless there are fundamental limits of this cooling method arising from the quantum mechanical behavior of the radiation field. In fact there are fluctuations around the mean value of the force, due to the discrete photon momentum transfer. These fluctuations give rise to diffusion due to the random directions of spontaneous emission of photons which leads to a broadening of the velocity distribution. Diffusion thus counteracts as a heating process.

The dipole force is independent of spontaneous emission. It arises from the induced dipole moment of the atom which according to the detuning of the laser draws the atom to high or low intensity regions of the field respectively. The steady state dipole force for the case of a motionless atom shows that atoms are attracted to high intensity regions for a red detuning, which means the laser frequency is bigger than the atomic transition frequency. The induced dipole moment is in phase with the laser field and the atom can decrease its energy in a field of high intensities - the atom is said to be a 'high field seeker'. For blue detuning, meaning the laser frequency is smaller than the atomic transition frequency and the induced dipole moment is 180° out of phase with the laser, the atoms are drawn to low intensity regions (low field seeker). To investigate the dipole force

independent of spontaneous emission, it is useful to operate with high detunings of the laser frequency from atomic transition frequency. One then finds that the dipole force is proportional to the gradient of the interaction energy $F = -\nabla H_{JC}$. In contrast to the radiation pressure force, the dipole force is not limited by the atomic linewidth. A very interesting feature of the dipole force is that it does not depend on the details of the structure of the atom, therefore it is applicable to molecules and even macroscopic objects. One takes advantage of this feature in so-called optical tweezers, where the objects of interest are trapped in the focus of a laser.

The idea now is to put the atom inside a cavity and study the interaction with the standing laser cavity field. Trapping and cooling properties are investigated in models of an atom strongly coupled to a single mode of a cavity standing laser field, where one gets alternative cooling methods. The conventional methods rely on cycles of optical pumping and spontaneous emission of a photon by the atom. Cavity cooling, however, does not require spontaneous emission by the atom, but cavity loss replaces this role. Cavity cooling can be related to a classical picture based on the notion of a refractive index. For strong atom-cavity coupling the intracavity intensity is strongly affected by the atomic motion. At a node of the cavity standing mode the atom is not coupled to the field. For resonant pumping the field intensity is large, whereas if the atom is at an antinode, it shifts the cavity out of resonance which leads to a small intracavity intensity. Since atomic excitation is low at all times, the minimum attainable temperature is not limited by the atomic linewidth but by the linewidth of the cavity, which can be much smaller, and temperatures below the Doppler limit can be reached.

This thesis is organized as follows:
The first chapter will be a review of the case of one atom interacting with a single mode of the cavity field. We will derive the Hamiltonian in the rotating frame approximation and calculate the Heisenberg equations of motion for the atom and the cavity field in the good cavity limit and by assuming the atomic excited state weakly populated. Finally with these approximations we will present the force acting on the atom.
In the second chapter we will enclose the backaction of the atom on the

field. This backaction will be modelled by a time-dependent potential, which we model as a harmonic potential with a time-dependent perturbation $\delta(t)$. At first we will check the quality of this model by keeping this perturbation time-independent in our calculations. Then we will calculate the time-dependent case and find numerical solutions for expectation values of the atom and the field amplitude.
The third chapter deals with the interaction between two atoms and the cavity field. We will apply the perturbative treatment of the previous chapter. And we will try a different ansatz and look for the eigensystem in order to find self-consistent eigenstates of the system.

Chapter 2

One atom interacting with a single cavity mode

2.1 Introduction

In the following chapter, we shortly review the case of one atom interacting with a single mode of a cavity standing field, including losses through decay of cavity photons. The Hamiltonian, describing a Jaynes-Cummings type interaction between the atom and a single mode of the standing laser field and the interaction of the cavity field with an external heat bath, will be calculated in a rotating frame to derive a master equation for the system. We will derive the Heisenberg-Langevin equations of motion for the atomic and mode operators, where we assume that the excited state of the atom is weakly populated (low saturation) and, as we consider the good cavity limit ($\kappa \gg g$), the atomic operators are treated as stationary values. With these approximations we will find an equation for the cavity field and the force acting on the atom.

2.2 The setup

We consider a 2-level atom with transition frequency ω_a and mass m coupled to a single mode of a high-finesse cavity with resonance frequency ω_c. The system is driven by a coherent laser field of frequency ω_L with pump rate η, injected through one of the mirrors of the cavity. For simplicity we neglect spontaneous emission of the atom, which is true for large atomic detunings $|\omega_a - \omega_L| \gg \gamma$, and restrict atomic motion to one dimension. The cavity field is being damped by coupling to the environment, photons

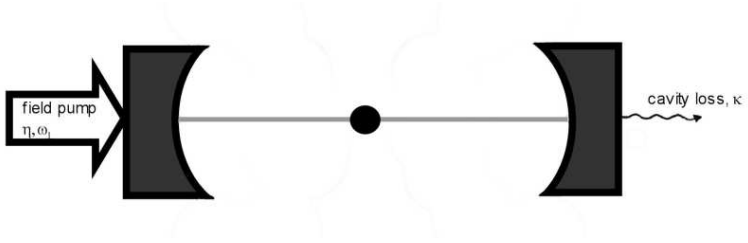

Figure 2.1: *One atom in a coherently pumped cavity*

decay with rate κ via the cavity mirror.
A schematic representation of the system is shown in Fig. 2.2

2.3 The Hamiltonian and the master equation

$$H_{sys} = H_a + H_c + H_{ac} + H_p \tag{2.1}$$

The Hamiltonian for this system contains several terms describing the motional and internal energy of the atom H_a, the self energy of the single mode cavity field H_c, the Jaynes-Cummings-type interaction between the atom and the cavity photons H_{ac} and the creation of cavity photons via the classical laser pump H_p.

$$H_a = \frac{p^2}{2m} + \hbar\omega_a\sigma^+\sigma^- \tag{2.2}$$

$$H_c = \hbar\omega_c a^\dagger a \tag{2.3}$$

$$H_{ac} = -i\hbar g(x)(\sigma^+ a - a^\dagger\sigma^-) \tag{2.4}$$

$$H_p = -i\hbar\eta(ae^{i\omega_L t} - a^\dagger e^{-i\omega_L t}) \tag{2.5}$$

σ^+ and σ^- denote the operators of the atomic polarization, $a^\dagger$ and a the creation and annihilation operators of the cavity field. The coupling between the atom at position x and the cavity mode is given by $g(x) = d\sqrt{\hbar\omega_c/2\epsilon\nu}u(x)$, where d is the atomic dipole moment, $u(x)$ the mode function and ν the mode volume. $g(x)$ gives the frequency of the energy exchange between atom and field at position x, known as Rabi-oscillations. For our standing wave cavity the mode function is $u(x) = cos(kx)$, with $k = 2\pi/\lambda$ being the cavity wave number.

In order to get rid of the explicit time dependence appearing in H_p we apply the unitary transformation U on the Hamiltonian H_{sys} so we get the new Hamiltonian H in a frame rotating with the pump frequency ω_L.

$$U = e^{i\omega_L t}(a^\dagger a + \sigma^+\sigma^-) \tag{2.6}$$

$$H = UH_{sys}U^\dagger + i\hbar(\frac{dU}{dt})U^\dagger \tag{2.7}$$

The new Hamiltonian in a rotating frame, where $\Delta_a = (\omega_L - \omega_a)$, $\Delta_c = (\omega_L - \omega_c)$ are the frequency shifts for the atom and the cavity mode, reads:

$$H = \frac{p^2}{2m} - \hbar\Delta_a\sigma^+\sigma^- - \hbar\Delta_c a^\dagger a - i\hbar g(x)(\sigma^+ a - a^\dagger\sigma^-) - i\hbar\eta(a - a^\dagger) \tag{2.8}$$

As mentioned above, in our model there is no spontaneous emission of photons by the atom, which is approximately true if the photon scattering rate Γ_0 - which describes the loss of photons via the atom - tends to zero.

$$\Gamma_0 = \frac{\gamma}{\Delta_a^2 + \gamma^2}\, g_0^2\bar{n}\ . \tag{2.9}$$

Here γ denotes the spontaneous emission rate of the atom (inverse linewidth), $g_0 = d\sqrt{\hbar\omega_c/2\epsilon\nu}$ is the coupling constant and $\bar{n}$ the average photon number.
This is true for large atomic detuning

$$\Delta_a \gg \gamma\ . \tag{2.10}$$

To complete the Hamiltonian we have to add another term describing the interaction of the cavity field with the environment, i.e. the annihilation of a cavity photon and the simultaneous creation of a photon in the reservoir

$$V_{cr} = \hbar\int d\omega\ \mathcal{B}(\omega)\ \big(b(\omega) + b^\dagger(\omega)\big)(a + a^\dagger)\ . \tag{2.11}$$

Where $b(\omega)$ are the bosonic field operators of the reservoir. They fulfill

$$\big[b(\omega), b^\dagger(\omega')\big] = \delta(\omega - \omega')\ . \tag{2.12}$$

The evolution of the system can be described in terms of the density operator

$$\dot{\rho}_{tot}(t) = -\frac{i}{\hbar}\big[H_{tot}(t), \rho_{tot}(t)\big]\ , \tag{2.13}$$

with the total Hamiltonian $H_{tot} = H_{sys} + V_{cr}$. We cannot solve this equation in general but with a few approximations the equation becomes solvable:

- The initial state of system and reservoir is assumed to be not entangled.
- Markoff-approximation: The autocorrelation time τ_c (i.e. the inverse bandwidth) of the reservoir is much smaller than any other characteristic time scale of the system.
- The density operator can be factorized for all times, which means the reservoir has to be large enough that the backaction of the system can be widely neglected.

$$\rho_{tot}(t) = \rho_{sys}(t) \otimes \mid vac >< vac \mid \tag{2.14}$$

By applying these approximations we know the density operator for the system for all times by tracing out the environment

$$\rho_{sys}(t) = Tr_{res}\big(\rho_{tot}(t)\big) = \rho(t) \ , \tag{2.15}$$

and we get the **master equation** [12]

$$\dot{\rho}(t) = -i\big[H_sys, \rho(t)\big] + \mathcal{L}\rho(t) \ . \tag{2.16}$$

The Liouville operator $\mathcal{L}$ includes the effects of cavity losses due to coupling to the environment.
The Hamiltonian and the Liouville operators in an interaction picture read

$$H = \frac{p^2}{2m} - \hbar\Delta_a\sigma^+\sigma^- - \hbar\Delta_c a^\dagger a - i\hbar g(x)(\sigma^+ a - a^\dagger\sigma^-) - i\hbar\eta(a - a^\dagger) \ , \tag{2.17}$$

$$\mathcal{L}\rho(t) = \kappa(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \ , \tag{2.18}$$

where κ denotes the decay rate for the cavity mode.

2.4 Heisenberg-Langevin equations and the low saturation limit

An equivalent way of treating this problem is via the Heisenberg-Langevin equations for the atomic and mode operators. The internal dynamics follow the equations

$$\dot{a} = (i\Delta_c - \kappa)a + g(x)\sigma^- + \eta + \xi(t) \tag{2.19}$$

$$\dot{\sigma^-} = i\Delta_a\sigma^- + g(x)\sigma_z a + \xi(t) \ . \tag{2.20}$$

where $\xi(t)$ denotes the noise operator [12]. Noise originates from fluctuations due to the coupling to the environment. The reservoir is assumed in the vacuum state (T=0), hence their expectation values vanish:

$$< vac|\xi|vac >= 0. \tag{2.21}$$

In the low saturation limit, which means that the atomic excited state is weakly populated, we can approximate the population inversion operator σ_z by $-\mathbf{1}$. Furthermore, in the good cavity limit ($\kappa \gg g$) the operators $\sigma^{\pm}$ evolve on a much faster timescale than the cavity field operators $a, a^{\dagger}$. Hence the atomic operators can be approximated by their stationary values, i.e. we set $\dot{\sigma}^{\pm} \approx 0$. This yields for the atomic dipole moment

$$\sigma^{-} = -i\frac{g(x)}{\Delta_a}\, a\ . \tag{2.22}$$

By inserting this in the equation for the cavity field, we get

$$\dot{a} = -\kappa a + i\big(\Delta_c - U_0 u^2(x)\big)a + \eta\ \ , \tag{2.23}$$

where $U_0 = \frac{g_0^2}{\Delta_a}$ is a measure for the shift of the cavity resonance frequency caused by the presence of the atom at position x. The atom shifts the cavity field into resonance with the pumping field if $U_0 \approx \Delta_c$.

2.5 Steady state dynamics, mean photon number and the force

As already mentioned in the previous section, the atom causes an effective cavity frequency shift which depends on the position distribution of the atom, i.e. the atom acts as a refractive index.
In order to get an overview of these dynamics it is useful to have a look at the steady state situation, i.e. the limit of a very slow atom such that a stationary value of the field according to the atom's position is obtained.

Taking the expectation value of the above equation for the cavity field (2.23) and substituting $< a >= \alpha$, one gets the equation for the coherent field amplitude α

$$\dot{\alpha} = -\kappa + i\big(\Delta_c - U_0 u^2(x)\big)\alpha + \eta\,. \tag{2.24}$$

By setting $\dot{\alpha} = 0$, the steady state solution can be easily calculated

$$\alpha_s = \frac{\eta}{-\kappa + i\big(\Delta_c - U_0 u^2(x)\big)}\ . \tag{2.25}$$

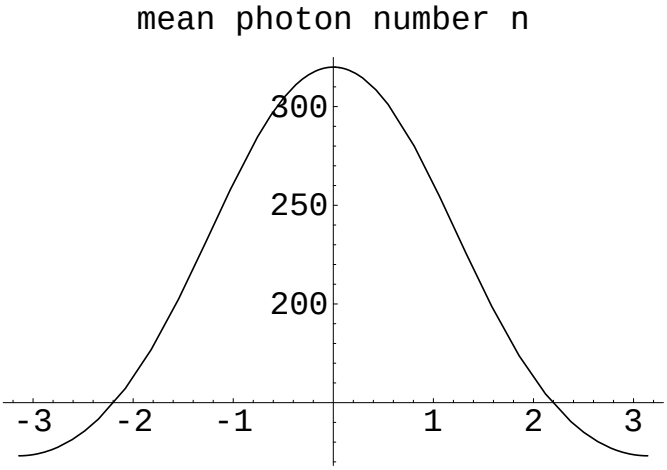

Figure 2.2: *The mean photon number $\bar{n} = |\alpha_s|^2$ as a function of the position distribution of the atom. a) $|\Delta_c| > |U_0|$, the cooling regime. The atom sitting at a node ($kx = 0$) is not coupled to the cavity mode and therefore intensity is highest. At the antinode ($kx = \pi$), where the atom is coupled to the cavity, it shifts the cavity to a higher frequency, this leads to a smaller intensity.*

From the above equation we can calculate the steady state mean photon number $\bar{n} = |\alpha_s|^2$:

$$\bar{n} = \frac{\eta^2}{\kappa^2 + \left(\Delta_c - U_0 u^2(x)\right)^2} \tag{2.26}$$

As printed in Fig. 2.5, the intensity is highest at the nodes of the cavity standing wave, where $kx = 0$, since the atom is not coupled to the cavity. The intensity decreases, however, if the atom approaches an antinode of the field mode, and is lowest at the antinode ($kx = \pi$), where the atom is strongly coupled to the cavity. It shifts the cavity to a higher frequency, which tunes the cavity out of resonance and leads to a small intensity.

The force operator F is proportional to the derivative of the momentum:

$$F = \dot{p} = -i[p, H] = -\nabla H \tag{2.27}$$

The only term in the Hamiltonian H, that will give a contribution, is the atom-photon interaction term H_{ac}

$$F = -\nabla H_{ac} \tag{2.28}$$

$$= -i\hbar\nabla g(x)(\sigma^+ a - a^\dagger \sigma^-) \tag{2.29}$$

If we do the adiabatic elimination $\dot{\sigma}^{\pm} \approx 0$ of the atomic operators σ^+ and σ^-, we get for their stationary values:

$$\begin{aligned} \sigma^+ &= i\frac{g(x)}{\Delta_a}\, a^\dagger \\ \sigma^- &= -i\frac{g(x)}{\Delta_a}\, a \end{aligned} \tag{2.30}$$

By inserting these equations in eq.3.2, we get an effective Hamiltonian

$$H_{eff} = \frac{p^2}{2m} - \hbar(\Delta_c - U_0 u^2(x))a^\dagger a - i\hbar\eta(a - a^\dagger)\ , \tag{2.31}$$

where $U_0 = \frac{g_0^2}{\Delta_a}$, as specified above.

Now we can find the force acting on a motionless atom by taking the directional derivative of the effective Hamiltonian H_{eff}

$$F(x) = -\hbar U_0\big(\nabla u^2(x)\big)a^\dagger a \tag{2.32}$$

When we take the mean value of the force, we can insert the mean photon number (2.26) for $< a^\dagger a >= |\alpha|^2$ and get

$$\begin{aligned} F(x) &= -\hbar U_0\big(\nabla u^2(x)\big)\bar{n} \\ &= -\hbar U_0\eta^2 \frac{\nabla u^2(x)}{\kappa^2 + \big(\Delta_c - U_0 < u^2(x) > \big)^2} \end{aligned} \tag{2.33}$$

Hence the force is proportional to the photon number and the gradient of the square of the mode function.
For $U_0 > 0$ the atom is drawn to field maxima (high field seekers), while $U_0 < 0$ corresponds to low field seeking atoms.

Chapter 3

One atom in a time-dependent harmonic trap

3.1 Introduction

In most applications of laser-atom interaction, the backaction of the atom(s) on the field can be largely neglected. For large enough atom-field detuning the influence of atoms on the field can be understood as a kind of dynamic refractive index. The field then forms a time dependent potential in the atomic Hamiltonian. In order to study this backaction, we consider an atom strongly coupled to a single mode of a driven high-Q optical resonator. The laser field is chosen far off resonance with the atomic transition frequency but close to cavity resonance, such that a coherent field can build up in the cavity but almost no spontaneous emission occurs. Thus, the atom basically acts as a moving refractive index, being large close to antinodes and showing almost no effect at a node of the cavity mode. In a first approximation the system can be related to a particle in a time-dependent harmonic trap, which is formed by the cavity field. For this we consider the atom well localized around a field antinode, where it sees a potential minimum. This allows to replace the optical potential locally by a harmonic potential. This suggests a mapping to a harmonic oscillator with time-dependent frequency. A method using parametrized oscillator eigenfunctions for any given $\omega(t)$ to solve such problems has been presented by Lewis and Riesenfeld [13]. In our case, however, we have no prescribed explicit form for the time-dependent frequency $\omega(t)$ and thus we will go for numerical solutions for the atomic expectation values and finding the field dynamics in parallel.

In this chapter, we investigate the coupled quantum dynamics of an atom in the cavity field by studying a system of an atom in a time-dependent harmonic trap. As a first step we start with a fixed harmonic potential and treat the time dependence by a time-dependent perturbation $\delta(t)x^2$. We will model the optical potential by a harmonic potential and rewrite the Hamiltonian for the atomic energy as a harmonic oscillator of frequency ω_0 to which the small perturbation $\delta(t)x^2$ is added. To check the validity of this perturbation expansion, we will keep δ constant and see if we get the same results as for a Hamiltonian with $\omega^2 = \omega_0^2 + \delta$. Then we will apply this model to a time-dependent perturbation $\delta(t)$ where we will find, that the system traps and localizes itself dynamically due to the backaction of the atom on the field.

3.2 The model

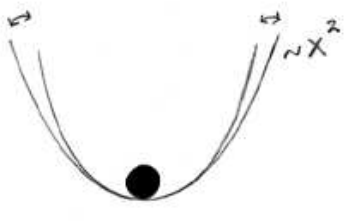

Figure 3.1: *The atom in the laser potential*

We consider the system discussed in the previous chapter and will now have a closer look on the backaction of the atom on the cavity field. The laser field is described by the mode function $u(x) = \cos(kx)$ and an amplitude α. As mentioned initiatively, the atom is supposed to be found close to an antinode of the field, so that the optical potential can be expanded in a Taylor series around the potential minimum (kx=0):

$$\cos^2(kx) \approx 1 - k^2x^2 \ . \tag{3.1}$$

This approximation models the laser potential locally as a harmonic oscillator potential. The interaction of the field and the atom sitting in this potential effects the depth of the laser potential, see Fig.3.2. We will model this effect by adding a small perturbation δ to the Hamiltonian for the harmonic oscillator. Let us assume that these changes are small compared to

the average field.
The Hamiltonian for the atom then reads

$$H(t) = H_0 + \delta(t)x^2 \,, \tag{3.2}$$

with

$$H_0 = \frac{p^2}{2m} + \frac{m\omega_0^2}{2}x^2 \,. \tag{3.3}$$

$H(t)$ thus describes a harmonic oscillator with time dependent frequency. We will perform this approximation by rewriting the Hamiltonian with frequency ω as harmonic oscillator H_0 to which the perturbation term δx^2 is added

$$\begin{aligned} H(t) &= \frac{p^2}{2m} + \frac{m\omega(t)^2}{2}x^2 \\ &= \frac{p^2}{2m} + \frac{m\omega_0^2}{2}x^2 + \underbrace{\frac{m}{2}(\omega^2 - \omega_0^2)}_{\delta(t)}\, x^2 \,. \end{aligned} \tag{3.4}$$

One can see, δ is proportional to the difference of the frequency ω and the harmonic oscillator eigenfrequency

$$\delta(t) = \frac{m}{2}(\omega(t)^2 - \omega_0^2) \,. \tag{3.5}$$

Rewriting ω in terms of the potential depth, which depends on the photon number $|\alpha|^2$, we get

$$\omega(t)^2 = \frac{2}{m}U_0|\alpha_t|^2 \,. \tag{3.6}$$

In order to calculate ω_0 in a way to keep $\delta(t)$ small, we choose the average field amplitude $|\alpha_0|^2$ in the unperturbed system ($\delta = 0$), and by taking $< x^2 >= 0$, as a reference field.
$\alpha_0 = \alpha|_{<x^2>=0}$

$$|\alpha_0|^2 = \frac{\eta^2}{\kappa^2 + (\Delta_c - U_0)^2} \,. \tag{3.7}$$

With this, we can calculate the Eigenfrequency in terms of the field amplitude

$$\omega_0^2 = \frac{2}{m}U_0|\alpha_0|^2 \,. \tag{3.8}$$

Inserting the equations for ω and ω_0 in equation (3.5) we get the time-dependent $\delta(t)$

$$\delta(t) = U_0(|\alpha_t|^2 - |\alpha_0|^2) \,. \tag{3.9}$$

3.3 Time-independent treatment

The idea is now to use the eigenstates of H_0 as a basis for the time-dependent calculation.
Nevertheless, at first we choose δ time-independent, in order to test the validity of the model and the calculations.
For a time-independent δ the results for the expectation values of position and momentum of the atom have to be the same as the solutions for the harmonic oscillator of frequency ω expressed in the basis ω_0. We will see in the further treatment, if we do our calculations in the basis of the harmonic oscillator $\mid \phi_i >$ the corrections to the expectation values for the atom will be proportional $c_n^* c_{n'}$ and therefore will have no effect. Of course, the additional term δx^2 will lead to a coupling of the equations for the coefficients $c_n(t)$, meaning that coefficients that initially are zero will start oscillating.

In order to calculate the expectation values for the system we expand the wavefunction $\mid \psi >$ in the Eigenbasis $\mid \phi_n >$ of the harmonic oscillator H_0

$$\mid \psi >= \sum_{n=0}^{\infty} c_n \mid \phi_n >, \tag{3.10}$$

and insert this ansatz in the Schrödinger equation, where we used the Hamiltonian (3.2).

$$< \phi_m \mid \quad \Big/ \quad i\mid \dot{\psi} > = H \mid \psi > \tag{3.11}$$

By doing the scalar product with $< \phi_m \mid$ we get the differential equation for the coefficients

$$\dot{c}_m = -i\omega m c_m - i c_n \delta \underbrace{< \phi_m \mid \widehat{x}^2 \mid \phi_n >}_{k_{mn}}, \tag{3.12}$$

where k_{mn} are the elements of the matrix

$$\mathbf{K} = \frac{1}{2m\omega} \begin{pmatrix} \ddots & 0 & \ddots & 0 & 0 \\ 0 & \ddots & 0 & \sqrt{n+1}\sqrt{n+2} & 0 \\ \ddots & 0 & 2n+1 & 0 & \ddots \\ 0 & \sqrt{n+1}\sqrt{n+2} & 0 & \ddots & 0 \\ 0 & 0 & \ddots & 0 & \ddots \end{pmatrix}. \tag{3.13}$$

So we have a set of coupled differential equations for the coefficients:

$$\dot{c}_m = -i\omega m c_m - i\frac{\delta}{2m\omega}\, k_{mn} c_n \ . \tag{3.14}$$

We find the numerical solutions and calculate the expectation values for the position $< x >$ and momentum $< p >$ of the atom

$$< x >=< \psi \mid x \mid \psi >= \sum_{n,n'} L_{nn'} c_n^* c_{n'} \ , \tag{3.15}$$

$$< p >=< \psi \mid p \mid \psi >= \sum_{n,n'} M_{nn'} c_n^* c_{n'} \ . \tag{3.16}$$

Where L and M are the matrices we find when calculating the expectation values $< \phi \mid x \mid \phi >$ and $< \phi \mid p \mid \phi >$ in the basis $\mid \phi >$ of the harmonic oscillator. As one can see in the above equations, the solutions for a time-independent δ are similar to the solutions for the harmonic oscillator H_0, since the real numbers $c_n^* c_{n'}$ do not contribute to the expectation values. But, as the additional term δx^2 causes a coupling of the coefficients, the coefficients that are initially zero start oscillating with amplitudes of order 10^{-4} till 10^{-15}, see Fig. 3.4. We choose the initial wavefunction as a superposition of the ground state and the first excited state of the unperturbed oscillator. The operating parameters are chosen: $\Delta_c = -10$ and $U_0 = -5$.

We compare the solutions of the method described above with the numerical solutions of the coupled differential equations that we derive from the time-independent Hamiltonian with frequency ω:

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 \tag{3.17}$$

where $\omega^2 = \frac{2}{m}\,\delta + \omega_0^2$.

$$\frac{d}{dt}(x^2) = -i[x^2, H] = \frac{1}{m}(xp + px) \tag{3.18}$$

$$\frac{d}{dt}(p^2) = -i[p^2, H] = -m\omega^2(xp + px) \tag{3.19}$$

$$\frac{d}{dt}(xp + px) = -i[xp + px, H] = 2\frac{p^2}{m} - 2m\omega^2 x^2 \tag{3.20}$$

$$\frac{d}{dt}(x) = -i[x, H] = \frac{p}{m} \tag{3.21}$$

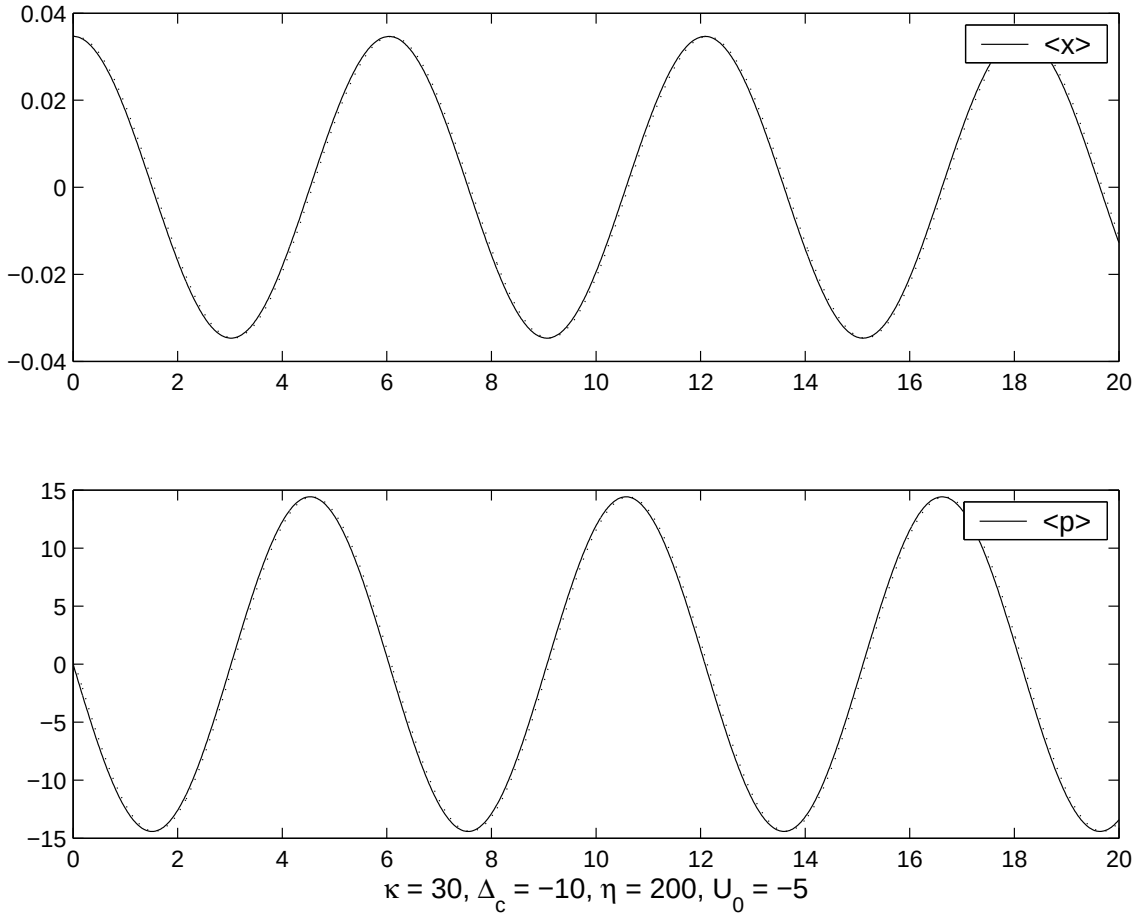

Figure 3.2: *Expectation values of the position x and momentum p of the atom in the cavity field for the perturbational method (solid line) and for the equations of motion with $\omega^2 = \frac{2}{m}\,\delta + \omega_0^2$ (dashed line). The parameters are $\kappa = 30, \Delta_c = -10, \eta = 200, U_0 = -5$.*

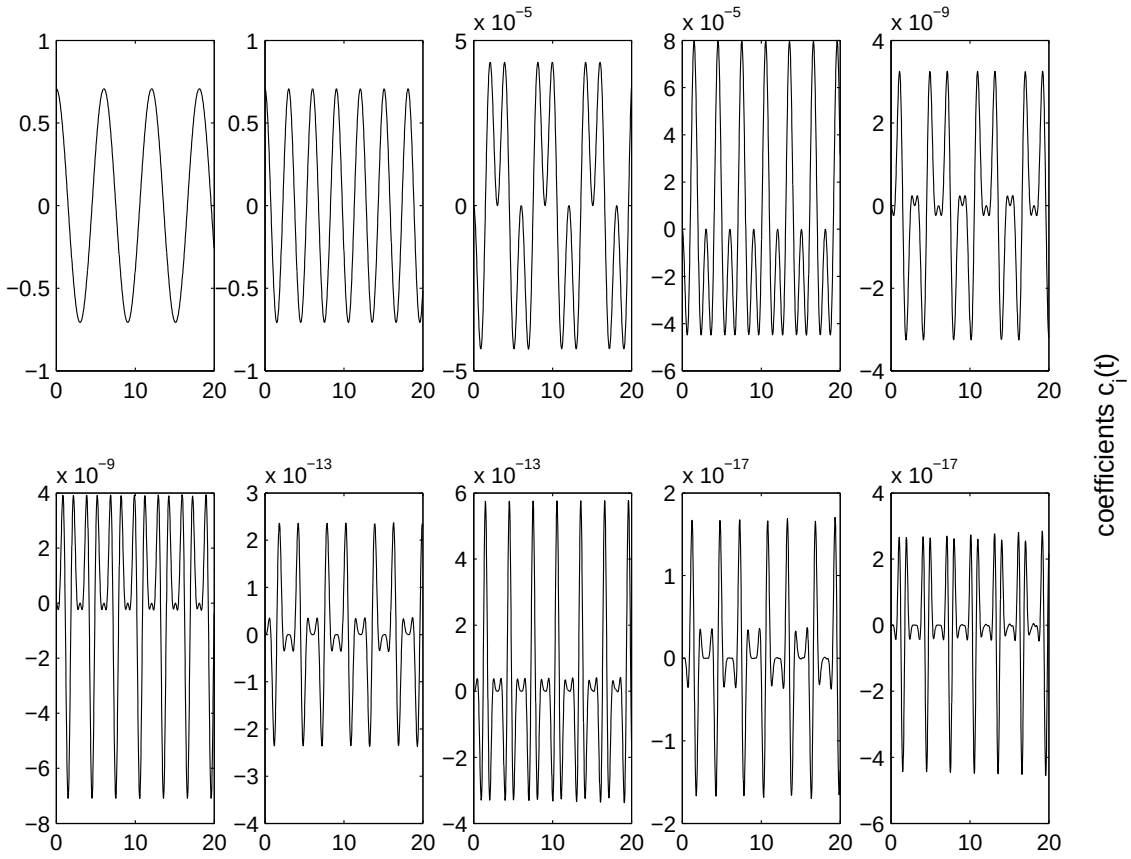

Figure 3.3: *N=10 coefficients, all of them initially zero, except for* $c_1 = c_2 = \frac{1}{\sqrt{2}}$*. The perturbation term* δx^2 *causes a coupling of the coefficients, such that coefficients that are initially zero start oscillating with amplitudes of order* 10^{-4} *till* 10^{-15}*. Parameters:* $\kappa = 30, \Delta_c = -10, \eta = 200, U_0 = -5$*.*

$$\frac{d}{dt}(p) = -i[p, H] = -m\omega^2 x \tag{3.22}$$

These coupled differential equations are fairly easy to solve numerically, independent of the form of the initial wavefunction.
We find the same curves when we plot the expectation values $< x >$ and $< p >$, as shown in Fig. 3.3.
Hence we see that our model works out right and we will use it to calculate the time-dependent Hamiltonian $H(t)$ in the next section and we will also apply it on the calculations for a system of two atoms inside a cavity, which will be performed in the next chapter.

3.4 Time-dependent perturbation $\delta(t)$

Now lets come to the full time dependent case and include the equation for α_t as well.
In the case of a time-dependent system, the Hamiltonian reads

$$H(t) = \frac{p^2}{2m} + \frac{m}{2}\omega(t)^2 x^2 \tag{3.23}$$

Again, we split the Hamiltonian into a harmonic part plus the time-dependent perturbation term

$$H(t) = H_0 + \delta(t)x^2 \tag{3.24}$$

with

$$\delta(t) = U_0(|\alpha_t|^2 - |\alpha_0|^2) \tag{3.25}$$

now being time-dependent.
Again we solve the equations

$$\dot{c}_m = -i\omega m c_m - i\frac{\delta(t)}{2m\omega}\, k_{mn} c_n \tag{3.26}$$

and

$$\dot{\alpha}_t = -\kappa + i\big(\Delta_c - U_0(1 - k^2 < x^2 >)\big)\alpha + \eta\,, \tag{3.27}$$

numerically. Note that Eq. (3.27) contains $< x^2 >$ which depends on the state of the atom described by the c_n.

Again, we choose the initial wavefunction as a superposition of the ground state and the first excited state and take a weak initial value for the field $\alpha(0) = 1$. The operating parameters $\Delta_c = -10$ and $U_0 = -5$ are chosen such that the system is in the cooling regime, the optical potential

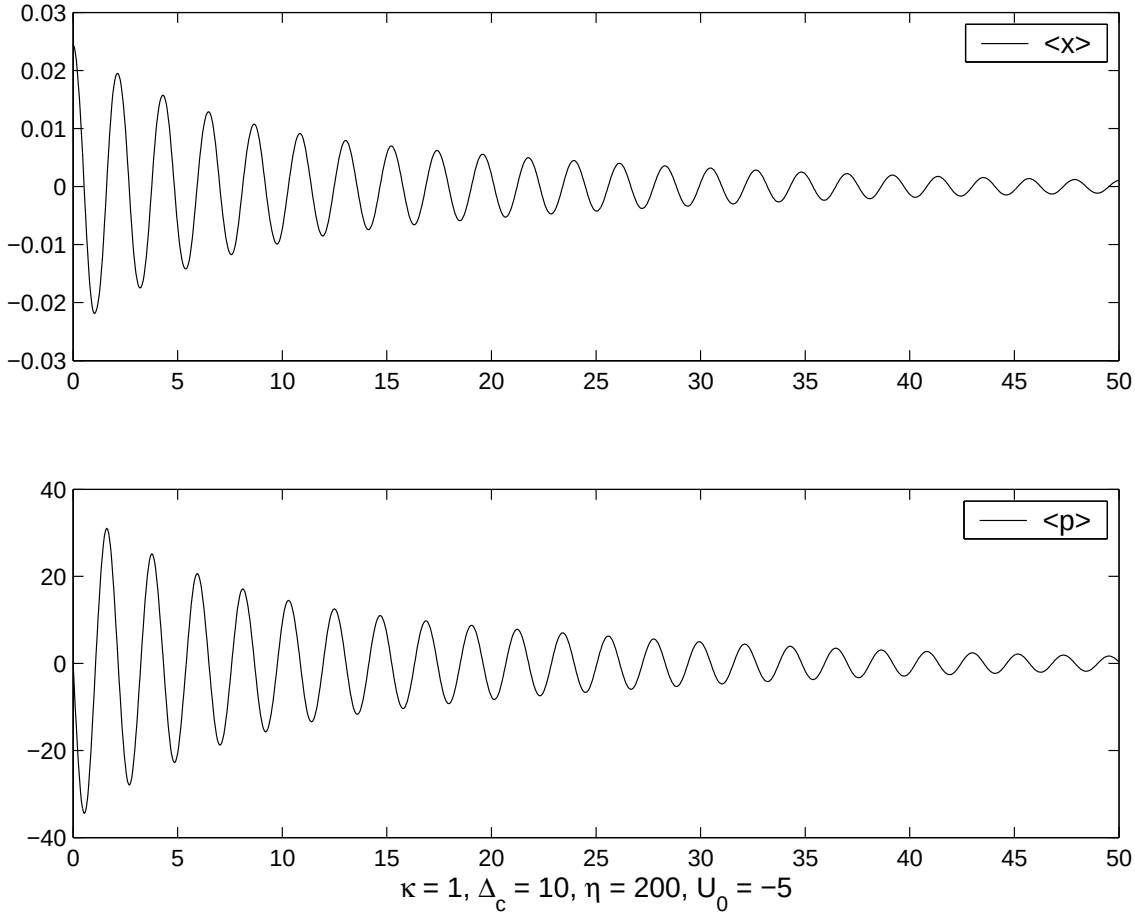

Figure 3.4: *Expectation values for the position and momentum of the atom. The expectation values* $< x >$ *and* $< p >$ *are damped. We chose* $\kappa = 1, \Delta_c = 10, \eta = 200, U_0 = -5$.

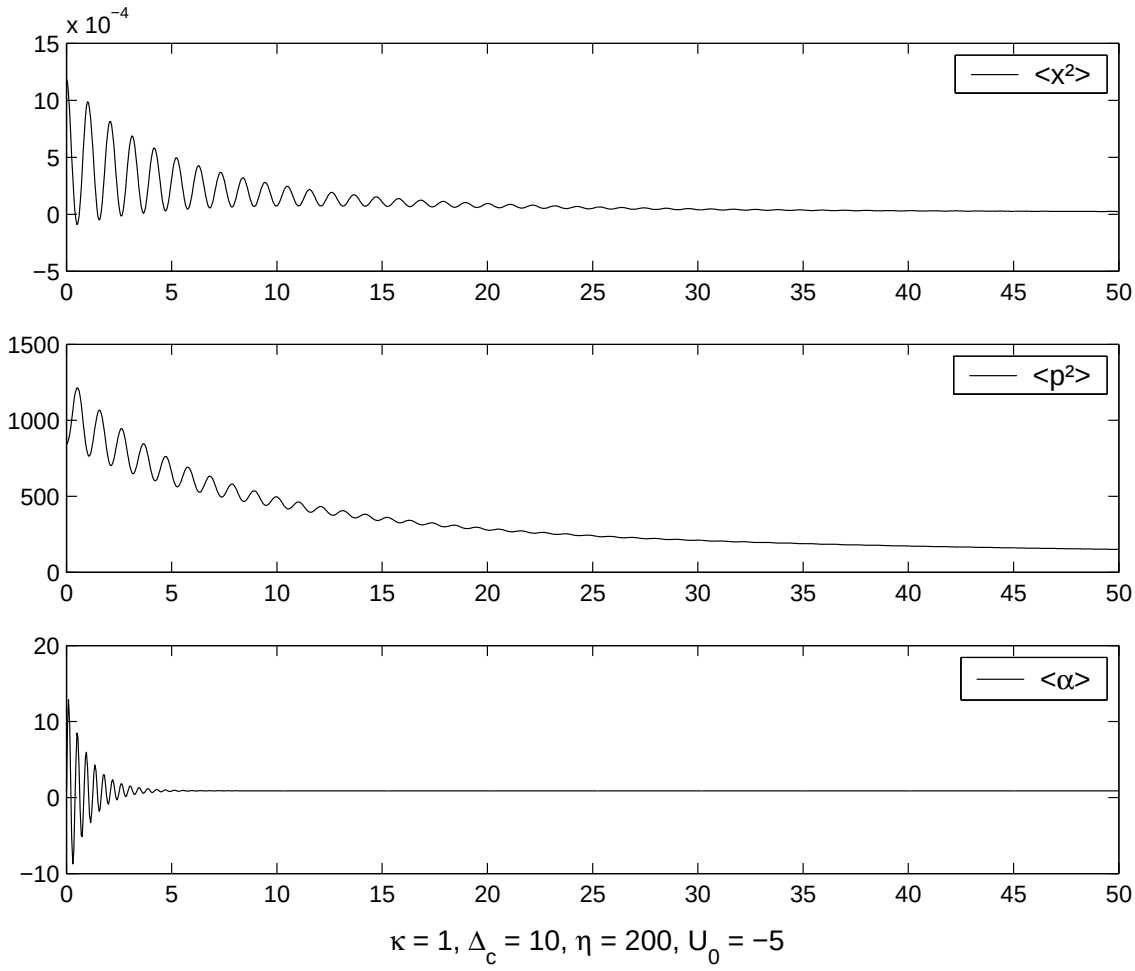

Figure 3.5: *Expectation values for the squared position and squared momentum of the atom, and the field amplitude α. The expectation values $< x^2 >$ and $< p^2 >$ are slightly damped. We chose $\kappa = 1, \Delta_c = 10, \eta = 200, U_0 = -5$.*

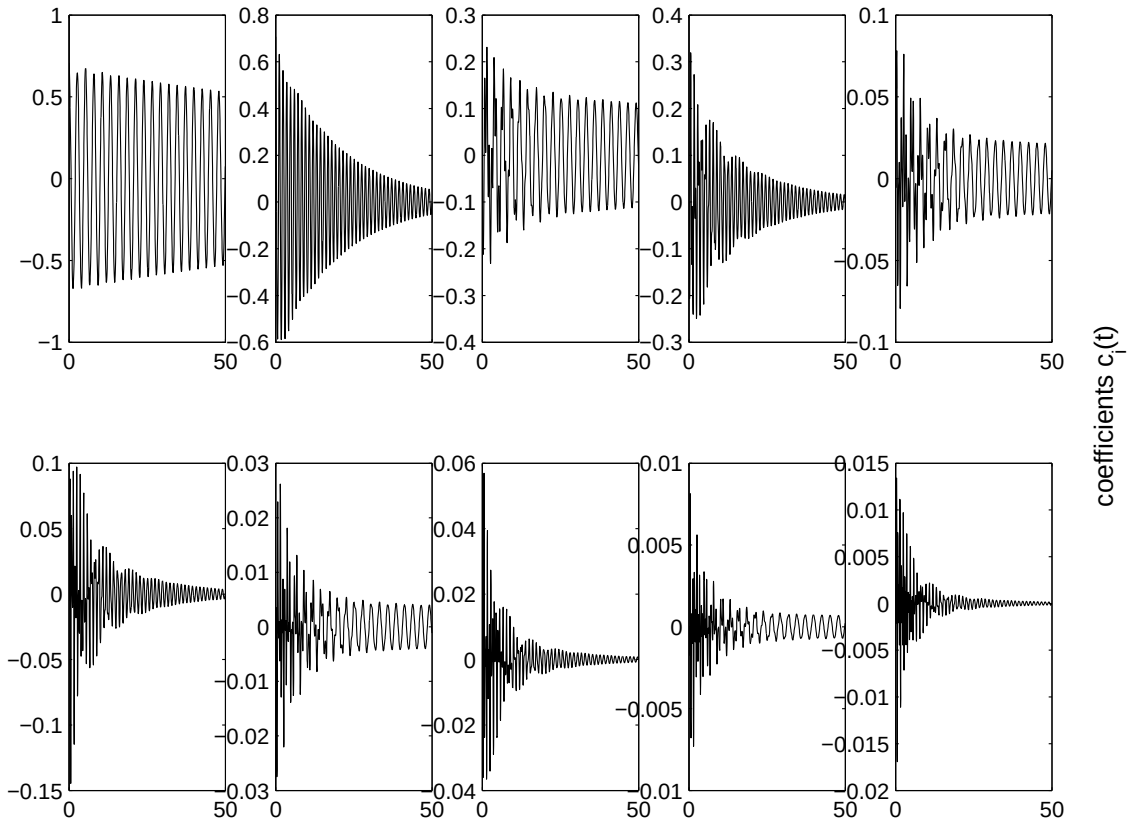

Figure 3.6: *The coefficients for the time-dependent calculation. We chose the parameters:* $\kappa = 1, \Delta_c = 10, \eta = 200, U_0 = -5$.

attracts the atom. Looking at Fig. 3.4, we see, that the the average potential energy proportional $< x^2(t) >$ and the cavity field amplitude $\alpha(t)$ tend to a stationary value, while the atomic motion is very weakly damped (Fig. 3.4). Hence, after some time the atom oscillates in self generated deep potential well with an almost constant field, i.e. the system dynamically traps and localizes itself around an antinode of the field. In contrast to classical mechanics, quantum mechanics allows states where $< x^2(t) >$ is constant while $< x(t) >$ is oscillating.

Cavity cooling in the quantum regime:
The cavity damping is used to extract energy from the system which allows dynamic localization of the wavefunction.
If $\Delta_a < 0$, which is equivalent to a red detuning from the transition frequency, the atom is pushed towards a field antinode, where it is strongly coupled to the field. When the atom moves away from a node, the intensity in the high-finesse cavity does not drop instantaneously, but leads to an increase of the energy stored in the field. The photons escape from the cavity and this extracts kinetic energy from the atom. The effect of acceleration of the atom when approaching the antinode is much smaller because the intracavity intensity there is small. This cooling process does not require atomic excitation and the lowest attainable temperature is not limited by atomic linewidth but by the cavity linewidth.

Chapter 4

Two interacting atoms inside a cavity

4.1 Introduction

We have seen in the one atom case, using the coupled atom-field dynamics, that the atom generates a potential well, where it traps itself dynamically.
Now if we put two atoms in the cavity together, they interact with the same field and although they are localized at distant positions they see each other through the cavity field. So even without direct coupling between the atoms one atom can influence the other because they communicate via the same field [14]. We additionally take into account a direct interaction between the atoms, i.e. if we allow collisions between them in our model. The interaction-potential is modelled by a delta-function multiplied with the parameter a which describes the interaction strength [15] between the atoms: $V(x_1, x_2) = a\,\delta(x_1 - x_2)$. This shape-independent approximation where the physical interaction is replaced by a point-like potential of zero range is well justified for ultracold atoms since their de Broglie wavelength is large enough to neglect the finer details of the interaction potential. One can relate the one-dimensional parameter a with the scattering length in 3 dimensions.

4.2 The Hamiltonian

We now are interested in the effects of two atoms in a cavity. The laser field is described by the mode function $u(x) = \cos(kx)$. The approximation $u^2(x) \approx 1 - k^2x^2$ near the potential minima we already introduced in

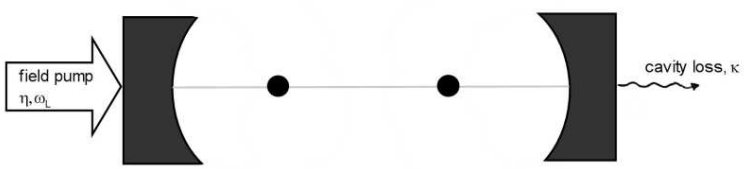

Figure 4.1: *Two atoms in a coherently pumped cavity*

the previous chapter, models the laser potential as a harmonic oscillator potential. The effect of the atoms on the cavity field shall be described by a time dependence of their harmonic potential as in the previous chapter. Again, we model this effect by adding a small perturbation δ to the Hamiltonian for the harmonic oscillator.
In addition, the atoms are considered to interact directly with each other via collisions. The delta-function $V(x_1, x_2) = a\ \delta(x_1 - x_2)$ represents the interaction-potential of this interaction, where a describes the interaction strength between the atoms.
As in the 1-atom case, we apply the low saturation approximation and introduce in a rotating frame the Hamiltonian for two atoms inside a cavity interacting with each other and interacting with a single mode of a driven laser field as

$$H_{eff} = H_0^{1,2} + \hbar\delta(t)k^2(x_1^2 + x_2^2) - \hbar(\Delta_c - 2U_0)|\alpha|^2 - i\hbar\eta(\alpha - \alpha^*) + \hbar V(x_1, x_2)\ . \tag{4.1}$$

The Hamiltonian consists of the motional and internal energy of each atom

$$H_0^{1,2} = \sum_{i=1,2} \frac{p_i^2}{2m} + \frac{m\omega_0^2}{2} x_i^2\ , \tag{4.2}$$

the perturbation term proportional $\delta(t)$, the atom-field interaction and the pumping term, which are independent of the atomic parameters, and the atom-atom interaction potential.

4.3 The differential equation for the coefficients in the harmonic eigenbasis

We expand the wavefunction for the 2-atoms system in the eigenbasis $\mid \phi_n(x_1)\phi_m(x_2) >$ of the unperturbed harmonic oscillator,

$$\mid \psi(t) > = \sum_{n,m=0}^{\infty} c_{n,m}(t) \mid \phi_n(x_1)\phi_m(x_2) > \ . \tag{4.3}$$

By inserting this wavefunction in the Schrödinger equation

$$| \dot{\psi}(t) >= -\frac{i}{\hbar} H_{eff} \mid \psi(t) > \tag{4.4}$$

we get the following differential equation for each coefficient $c_{kl}(t)$

$$\dot{c}_{kl}(t) = -i\big(\omega_0(k+l)c_{kl} - i\mathcal{E}_\alpha c_{kl} + \delta(t)\mathcal{A} + \mathcal{U}\big) \ , \tag{4.5}$$

As $\mathcal{E}_\alpha$ is independent of n, m, k, l, it gives only a global phase shift. So we will omit this term in the further calculations.

$$\mathcal{E}_\alpha = i(\Delta_c - 2U_0)|\alpha|^2 - \eta(\alpha - \alpha^*) \tag{4.6}$$

$\mathcal{A}$ contributes to the perturbation term, which is calculated via the expectation values for the unperturbed system

$$\mathcal{A} = \sum_{n,m} k^2 c_{nm}(t) < \phi_k^1 \phi_l^2 | (x_1^2 + x_2^2) | \phi_n^1 \phi_m^2 > \ , \tag{4.7}$$

which yields

$$\begin{aligned} \mathcal{A} &= \frac{k^2 x_0^2}{2} \sum_{n,m} c_{nm}(t)(\delta_{lm} x_{kn} + \delta_{kn} x_{lm}) \\ &= \frac{k^2 x_0^2}{2} \sum_{n,m} (c_{nl}(t) x_{kn} + c_{km}(t) x_{lm}) \ . \end{aligned}$$

The x_{ij} denote the matrix elements we get when calculating the expectation values for the harmonic system and k^2 is the wave vector of the field mode.
As the first nontrivial case we restrict ourselves to $n, m = 1, 2$. Hence, we only take into account the first two levels in the harmonic oscillator potential for each atom. The wavefunction for the system then reads

$$\begin{aligned} | \psi >= \ & c_{11} \mid \phi_0(x_1)\phi_0(x_2) > + c_{12} \mid \phi_0(x_1)\phi_1(x_2) > + \\ & c_{21} \mid \phi_1(x_1)\phi_0(x_2) > + c_{22} \mid \phi_1(x_1)\phi_1(x_2) > \ . \end{aligned} \tag{4.8}$$

The coefficients describe both atoms in the ground state (c_{11}), one atom in the ground state, one atom in the first excited state and vice versa (c_{12}, c_{21}) and both atoms in the first excited state (c_{22}).
Taking into account that we are dealing with only 2 states of the atoms, we get

$$\mathcal{A} = \frac{k^2 x_0^2}{2}(2x_{11}c_{11} + 2x_{22}c_{22} + (x_{12} + x_{21})(c_{12} + c_{21})) \ , \tag{4.9}$$

which yields the matrix

$$\mathbf{A} = \frac{k^2 x_0^2}{2} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix} . \tag{4.10}$$

As one can see, the matrix is diagonal and does not couple different coefficients c_{kl}. Hence, the perturbation thus does not lead to vibrational transitions, but only to time dependent phase shifts.

The last term, describing the direct interaction between the two atoms, reads

$$\mathcal{U} = \sum_{n,m} c_{nm}(t) < \phi_k^1 \phi_l^2 | V(x_1, x_2) | \phi_n^1 \phi_m^2 > \ . \tag{4.11}$$

We model the potential by the delta function described above

$$V(x_1, x_2) = a\delta(x_1 - x_2) \ , \tag{4.12}$$

with a being the interaction parameter of the two atoms.
Let us now have a deeper look at the scalar product in equation (4.11). We integrate the delta function and are left with the 1-dimensional integral

$$\begin{aligned} < \phi_k^1 \phi_l^2 | \delta(x_1 - x_2) | \phi_n^1 \phi_m^2 > &= \int\int dx^1 dx^2 \phi_k^{1*} \phi_l^{2*} \delta(x_1 - x_2) \phi_n^1 \phi_m^2 \\ &= \int dx^1 \phi_k^* \phi_l^* \phi_n \phi_m =: U_{klnm} \ . \end{aligned} \tag{4.13}$$

The two possible states for each atom are the ground state

$$\phi_0(x) = \sqrt[4]{\frac{m\omega}{\hbar\pi}} e^{-\frac{1}{2}(\frac{x}{x_0})^2} \ , \tag{4.14}$$

and the first excited state of the oscillator potential

$$\phi_1(x) = \frac{2x}{x_0} \sqrt[4]{\frac{m\omega}{\hbar\pi}} e^{-\frac{1}{2}(\frac{x}{x_0})^2} \ . \tag{4.15}$$

If we are looking at the integrals $\mathcal{U}_{klnm}$, we can exclude those combinations of ϕ_i 's that have an impair part of antisymmetric wavefunctions ϕ_2, i.e. $\mathcal{U}_{1222}, \mathcal{U}_{1112}, \mathcal{U}_{2122}$, etc. , because integration of an odd function gives zero.
We can separate the treatment of all other $\mathcal{U}_{klnm}$ into solving 3 different

integrals:
case I: all ϕ are in the ground state

$$\mathcal{U}_{1111} = \frac{m\omega}{\hbar\pi} \int dx e^{-2(\frac{x}{x_0})^2} \,, \tag{4.16}$$

case II: all ϕ are in the $1^s t$ excited state

$$\mathcal{U}_{2222} = \frac{m\omega}{\hbar\pi} \int dx (2\frac{x}{x_0})^4 e^{-2(\frac{x}{x_0})^2} \,, \tag{4.17}$$

case III: two ϕ in ground state, two ϕ in the $1^s t$ excited state

$$\mathcal{U}_{mixed} = \frac{m\omega}{\hbar\pi} \int dx (2\frac{x}{x_0})^2 e^{-2(\frac{x}{x_0})^2} \,, \tag{4.18}$$

where the index 'mixed' = 1212, 2121, 1122, 2211, 1221, 2112.
The integrals are of the form:

$$\int_{-\infty}^{\infty} e^{-px^2} x^{2q} dx = \frac{(2q-1)!!}{(2p)^q} \sqrt{\frac{\pi}{p}} \,. \tag{4.19}$$

In our case $p = 1, q = 0, 1, 2$. By substituting $\xi = \sqrt{2}\frac{x}{x_0} \longrightarrow dx = \frac{x_0}{\sqrt{2}}\, d\xi$ and writing the constant $\frac{m\omega}{\hbar}$ in terms of x_0 we get

$$\begin{aligned} \frac{2^q}{x_0\pi\sqrt{2}} \int_{-\infty}^{\infty} d\xi e^{-\xi^2} \xi^{2q} &= \frac{2^q}{x_0\pi\sqrt{2}} \frac{(2q-1)!!}{2^q} \sqrt{\pi} \\ &= \frac{(2q-1)!!}{x_0\sqrt{2\pi}} \,. \end{aligned} \tag{4.20}$$

case I: $q = 0 \quad \int_{-\infty}^{\infty} d\xi e^{-\xi^2} = \sqrt{\pi}$

$$\mathcal{U}_{1111} = \frac{1}{x_0\sqrt{2\pi}} \,. \tag{4.21}$$

case II: $q = 2 \quad \int_{-\infty}^{\infty} d\xi \; e^{-\xi^2} \; \xi^4$

$$\mathcal{U}_{2222} = \frac{3}{x_0\sqrt{2\pi}} \,. \tag{4.22}$$

case III: $q = 1 \quad \int_{-\infty}^{\infty} d\xi \; e^{-\xi^2} \; \xi^2$

$$\mathcal{U}_{mixed} = \frac{1}{x_0\sqrt{2\pi}} \,, \tag{4.23}$$

We resume the $\mathcal{U}_{klnm}$ in matrix form and obtain

$$\mathbf{U} = \frac{1}{x_0\sqrt{2\pi}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} . \tag{4.24}$$

The matrix U shows off-diagonal terms and appears in a block form.
The off diagonal terms cause a coupling of the differential equations for the coefficients. The block form of the matrix lets us assume that there are actually two independent systems, meaning coefficients c_{11} and c_{22} are coupled and c_{12} and c_{21} are coupled.
We will see later that this assumption is right (4.4.1).

4.4 Numerical solutions

Having calculated the scalar products of the different terms in the equation for the coefficients, we express the 2-atom Schrödinger equation in matrix form

$$\dot{\vec{c}} = -i[\omega_0 S + \delta(t)A + aU]\vec{c}\,, \tag{4.25}$$

where

$$\mathbf{S} = \sum_{k,l=1,2} (k+l) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{4.26}$$

is the matrix arising from the harmonic oscillator term H_0.
To complete the set of differential equations we need the equation for the cavity field, which depends on the atomic motion:

$$\dot{\alpha} = [-\kappa + i\big(\Delta_c - U_0(2 - k^2 < x^2 >)\big)]\alpha + \eta \tag{4.27}$$

where $< x^2 >$ now is the expectation value of the squared atomic positions which are proportional to the atomic potential energies:

$$< x^2 >=< x_1^2 > + < x_2^2 >= \sum_{nm} |c_{nm}|^2 (x_{nn} + x_{mm})\,. \tag{4.28}$$

These expectation values can be calculated from the solution of the Schrödinger Equation (4.25).

Again as a reference state for detuning the unperturbed basis, we choose α_0 by setting $< x^2 >= 0$
$\alpha|_{<x_1^2>+<x_2^2>=0} = \alpha_0$

$$|\alpha_0|^2 = \frac{\eta^2}{\kappa^2 + (\Delta_c - 2U_0)^2} \, . \tag{4.29}$$

And the equation for $\delta(t)$ according to the previous chapter reads

$$\delta(t) = \omega_t^2 - \omega_0^2 = \frac{2}{m} U_0(|\alpha_t|^2 - |\alpha_0|^2) \, . \tag{4.30}$$

We numerically solve the coupled differential equations

$$\dot{\vec{c}} = -i[\omega_0 S + \delta(t) A + aU]\vec{c} \tag{4.31}$$

and

$$\dot{\alpha} = [-\kappa + i(\Delta_c - U_0(2 - k^2 < x^2 >))]\alpha + \eta \, . \tag{4.32}$$

Looking at the plots, we see that the two atoms exchange their energy periodically. The cavity field α tends to its stationary value after some time and the squared expectation values for position $< x^2 >$ and momentum $< p^2 >$ are slightly oscillating around their steady state values.
The plots show the coupled atom-field dynamics for different values of the interaction strength $a = 0.001$, $a = 0.01$, $a = 0.05$ and $a = 0.1$. For a small interaction strength ($a = 0.001$ to 0.01) the field dynamics relax after several oscillations and reach a stationary value. The atoms' position and momentum expectations oscillate with periodically shifted amplitudes and the period at which the atoms exchange the energy seems to be much longer compared to the case of bigger interaction strengths a.
With growing values of a the field amplitude α does not reach the stationary value but it oscillates around it sinusoidally. The expectation values for the squared atomic position $< x^2 >$ and momentum $< p^2 >$ oscillate around their steady state values too.

4.4.1 Initial conditions

It is interesting to see that for certain initial conditions of the coefficients c_{ik} in the wavefunction $\mid \psi >$ the system stays stationary.

$$\begin{aligned} \mid \psi > = \; & c_{11} \mid \phi_0(x_1)\phi_0(x_2) > + c_{12} \mid \phi_0(x_1)\phi_1(x_2) > + \\ & c_{21} \mid \phi_1(x_1)\phi_0(x_2) > + c_{22} \mid \phi_1(x_1)\phi_1(x_2) > \end{aligned} \tag{4.33}$$

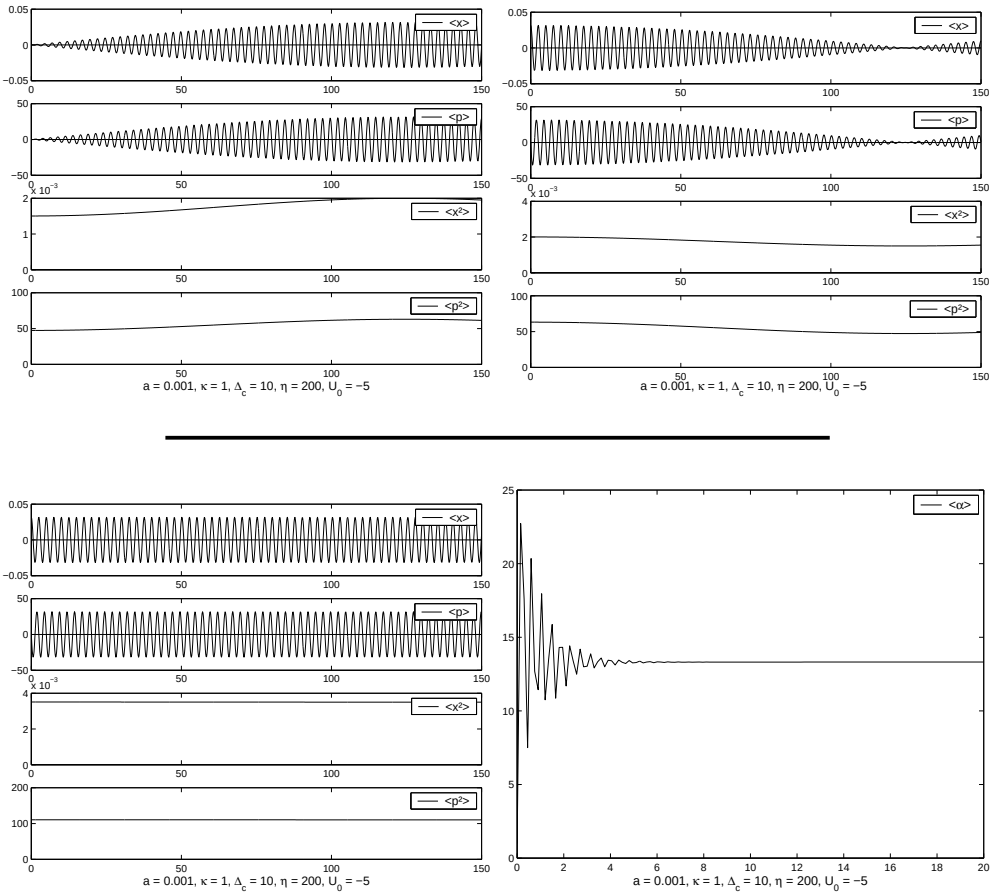

Figure 4.2: *Expectation values for the two atoms in the cavity (upper picture), the combined expectation values for both atoms and the cavity field α (lower picture) with an interaction strength $a = 0,001$ and operating parameters $\kappa = 1, \Delta_c = 10, \eta = 200, U_0 = -5$.*

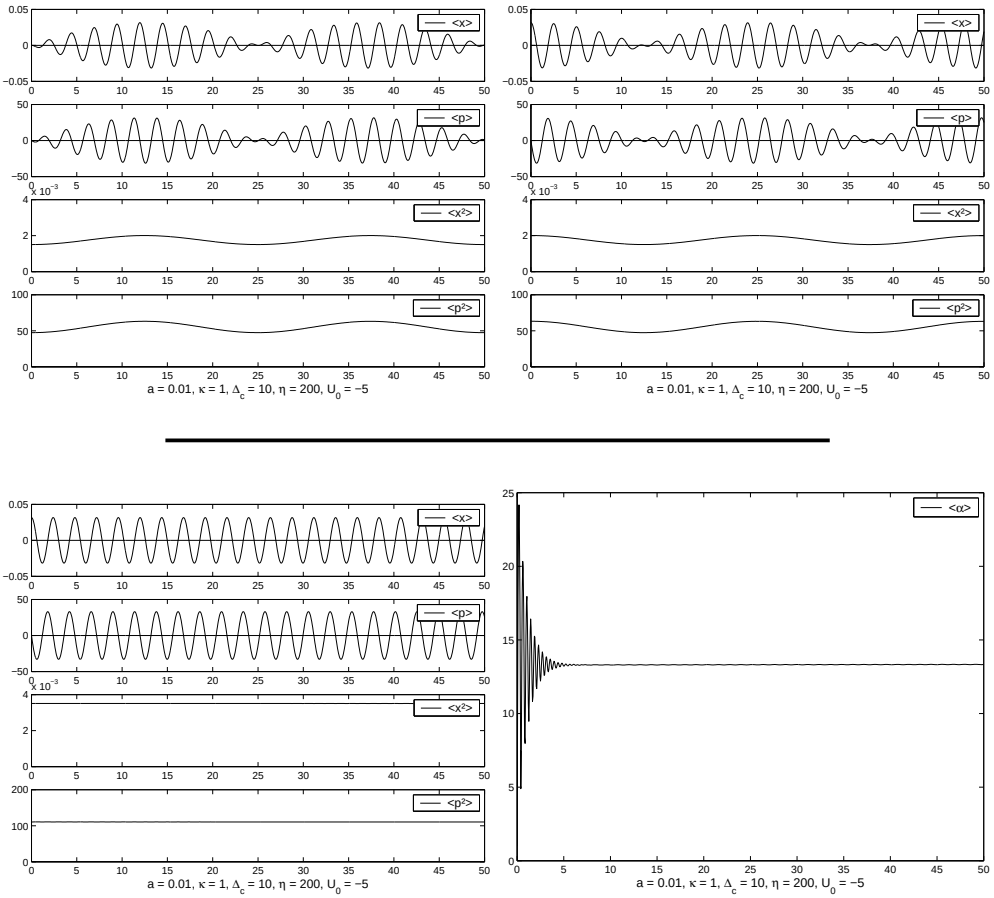

Figure 4.3: *Expectation values of the position x and momentum p of the first and second atom in the cavity field (upper picture) and expectation values of the position x and momentum p of both atoms in the cavity field and the expectation value of the cavity field α (lower picture). Parameters: $\kappa = 1, \Delta_c = 10, \eta = 200, U_0 = -5$, interaction strength $a = 0,01$.*

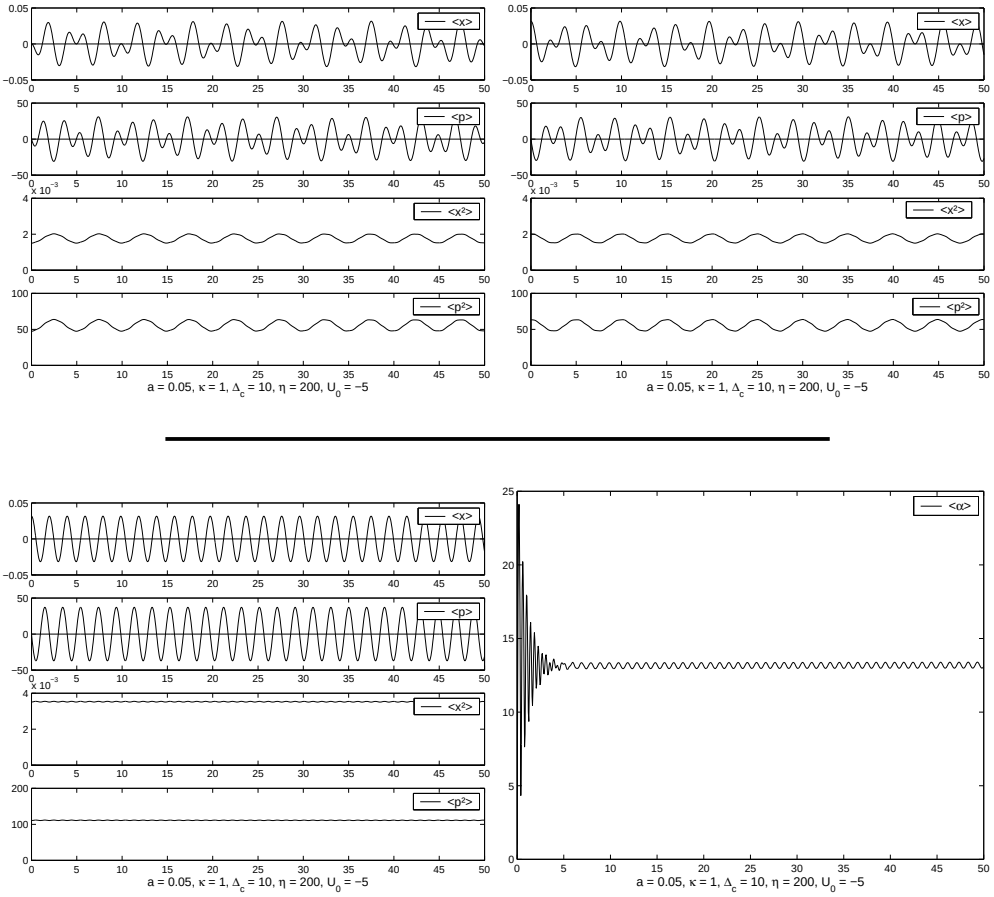

Figure 4.4: *Expectation values for the two atoms in the cavity (upper picture), the combined expectation values for both atoms and the cavity field* α *(lower picture) with an interaction strength* $a = 0,05$ *and operating parameters* $\kappa = 1, \Delta_c = 10, \eta = 200, U_0 = -5$.

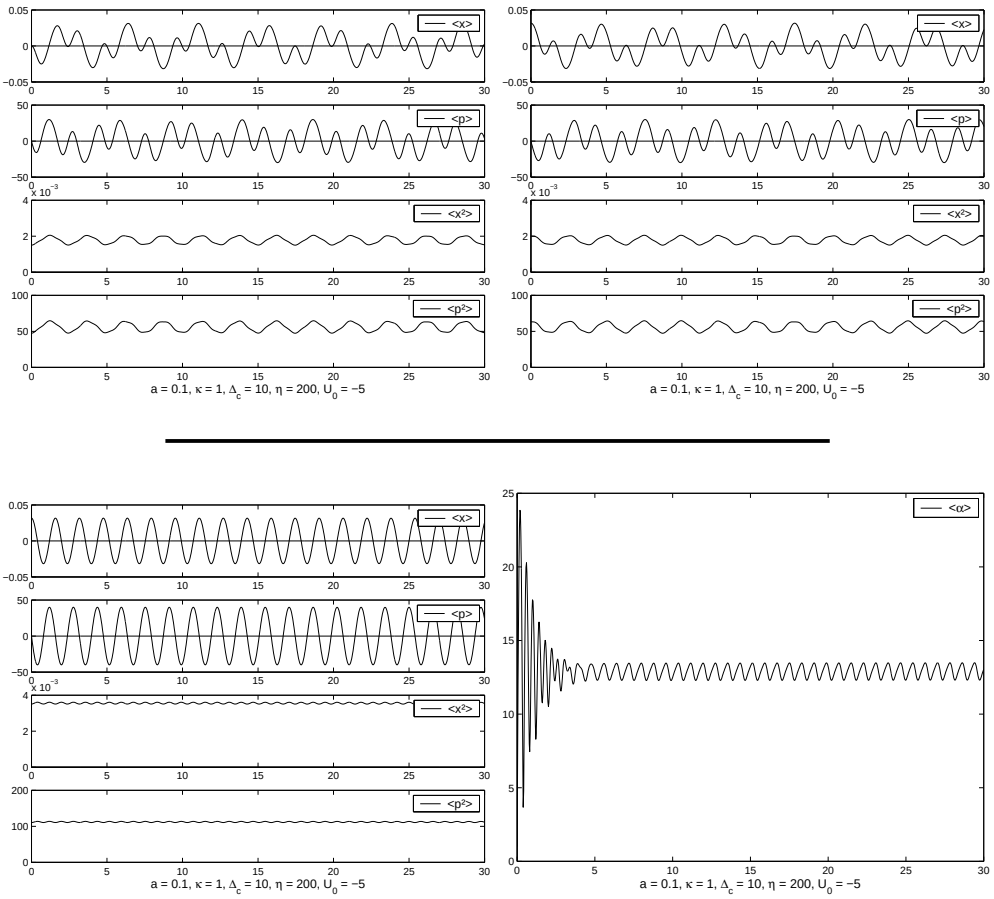

Figure 4.5: *Expectation values for the two atoms in the cavity (upper picture), the combined expectation values for both atoms and the cavity field* α *(lower picture) with an interaction strength* $a = 0,1$ *and operating parameters* $\kappa = 1, \Delta_c = 10, \eta = 200, U_0 = -5$.

As an example, if the system initially is in a superposition of both atoms in the ground state and both atoms in the first excited state ($c_{11} = c_{22} = \frac{1}{\sqrt{2}}$, $c_{12} = c_{21} = 0$), i.e.

$$\mid \psi > = \frac{1}{\sqrt{2}} \mid \phi_0(x_1)\phi_0(x_2) > + \frac{1}{\sqrt{2}} \mid \phi_1(x_1)\phi_1(x_2) >, \tag{4.34}$$

the expectation values for the position and momentum remain stationary while those for the potential and kinetic energies of each atom are slightly oscillating. The field is damped after some oscillations but it keeps oscillating with bigger a amplitude the bigger the value of the atomic interaction strength a.
If we choose the superposition of one atom being in the ground state, the second in the first excited state and vice versa, i.e. $c_{12} = c_{21} = \frac{1}{\sqrt{2}}$, $c_{11} = c_{22} = 0$,

$$\mid \psi > = \frac{1}{\sqrt{2}} \mid \phi_0(x_1)\phi_1(x_2) > + \frac{1}{\sqrt{2}} \mid \phi_1(x_1)\phi_0(x_2) > , \tag{4.35}$$

the expectation values for the position and momentum and the squared position and squared momentum remain stationary. The field is oscillating and reaches a stationary value after some time independent of the interaction strength a between the atoms.

Recalling the matrix U, which describes the direct interaction between the two atoms,

$$\mathbf{U} = \frac{1}{x_0\sqrt{2\pi}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} , \tag{4.36}$$

we see that the observations described above are a consequence of the special shape of the matrix U. As we already assumed in section 4.3, if we take either c_{11}, c_{22} non-zero or c_{12}, c_{21} non-zero for the initial conditions, the cases described above will occur because the differential equations for $\dot{c_{11}}$ and $\dot{c_{22}}$ such as $\dot{c_{12}}$ and $\dot{c_{21}}$ are not independent.
On the other hand, if we initially take any mixture of the two sets, say $c_{11} = c_{12} = \frac{1}{\sqrt{2}}$, $c_{22} = c_{21} = 0$,

$$\mid \psi > = \frac{1}{\sqrt{2}} \mid \phi_0(x_1)\phi_0(x_2) > + \frac{1}{\sqrt{2}} \mid \phi_0(x_1)\phi_1(x_2) > , \tag{4.37}$$

the atomic position and momentum expectation values oscillate with shifted periodically varying amplitudes, as shown in fig 4.4, where we chose the

above initial condition.

There is another interesting feature if we are looking at the matrix A arising from the perturbation term. As we already pointed out in section 4.3, the matrix A is diagonal. We compare it with the matrix K (Eq.(3.13), section 3.3) in the perturbation term in the one atom case, where the matrix is of the form

$$\mathbf{K} = \frac{1}{2m\omega} \begin{pmatrix} \ddots & 0 & \ddots & 0 & 0 \\ 0 & \ddots & 0 & \sqrt{n+1}\sqrt{n+2} & 0 \\ \ddots & 0 & 2n+1 & 0 & \ddots \\ 0 & \sqrt{n+1}\sqrt{n+2} & 0 & \ddots & 0 \\ 0 & 0 & \ddots & 0 & \ddots \end{pmatrix} . \quad (4.38)$$

in contrast to the matrix K in the perturbation term for the one-atom case, which shows entries in the main diagonal and in the two second diagonals. In the one-atom case this matrix causes the coupling of the differential equations for the coefficient with the effect, that coefficients that initially are zero start oscillating. The coupling does not effect the expectation values for the atom and field.
Here, in the case of two atoms in the cavity, if we consider no interaction between the atoms, i.e. a=0, we make the following observations:
If initially there is a superposition of both atoms in the ground state and both atoms in the first excited state the system will remain steady, same for the superposition of one atom in the ground state, one in the first excited and vice versa. In the latter case this is also true for $a \neq 0$.
For the remaining possible initial conditions only one atom remains steady, for the other one we find harmonically oscillating expectation values $< x >$ and $< p >$. As an example, we initially take the first atom in the ground state, the second in the first excited state and both atoms in the first excited state. We will see, only the first atom will have oscillating expectation values for $< x >$ and $< p >$, the second will remain steady.

4.5 Self consistent eigenstates

Let us now try to investigate the stationary states of the system in more detail.
We are interested in an analytical treatment of the problem, so we consider

the algebraic method of finding stationary states of the atom field dynamics. The idea is to look for the eigensystem of the matrix equation (4.25) for the coefficients in order to find a solution of the form $c(t) = v_i e^{-ipt}$ spanned by the eigenvectos v_i. Using such coefficients in Eq. (4.28) the calculation of the expectation values will yield constant values since the time dependence cancels. When we look for the corresponding field α which depends on $< x^2 >$, thus will be a constant in this case. By these means we then found a stationary state of the total system. For self consistency we try to find an η such that it matches the found α and δ.

To start, we first calculate the eigensystem of the matrix M appearing in the differential equation for the coefficients

$$\dot{\vec{c}} = -iM\vec{c}\,, \tag{4.39}$$

where

$$M = \left[\omega_0 S + \delta(t)A + aU\right]. \tag{4.40}$$

Introducing the abbreviations $u = \frac{k^2 x_0^2}{2}$ and $v = \frac{1}{x_0\sqrt{2\pi}}$ the matrix M reads

$$M = \begin{pmatrix} 2\omega_0 + 6\delta u + av & 0 & 0 & av \\ 0 & 3\omega_0 + 8\delta u + av & av & 0 \\ 0 & av & 3\omega_0 + 8\delta u + av & 0 \\ av & 0 & 0 & 4\omega_0 + 10\delta u + 3av \end{pmatrix}$$

By diagonalizing the matrix M we find the eigenvalues

$$p_1 = 3\omega_0 + 8\delta u \tag{4.41}$$

$$p_2 = 3\omega_0 + 8\delta u + 2av \tag{4.42}$$

$$p_3 = 3\omega_0 + 8\delta u + 2av - \sqrt{(\omega_0 + 2\delta u + av)^2 + a^2 v^2} \tag{4.43}$$

$$p_4 = 3\omega_0 + 8\delta u + 2av + \sqrt{(\omega_0 + 2\delta u + av)^2 + a^2 v^2} \tag{4.44}$$

where p_1 does not depend on a, p_2 shows a linear, p_3 and p_4 a hyperbolic dependence on a, see fig 4.5. The plot shows also the dependence of the eigenvalues for a negative interaction strength $a < 0$, which corresponds to an attractive interaction between the atoms, and a positive interaction strength $a > 0$, resembling a repulsive interaction. For $a < 0$ the energy levels are shifted downward and for $a > 0$ levels are shifted upward. In the case $a = 0$, meaning no interaction between the atoms, the eigenvalues p_1 and p_2 are degenerate.

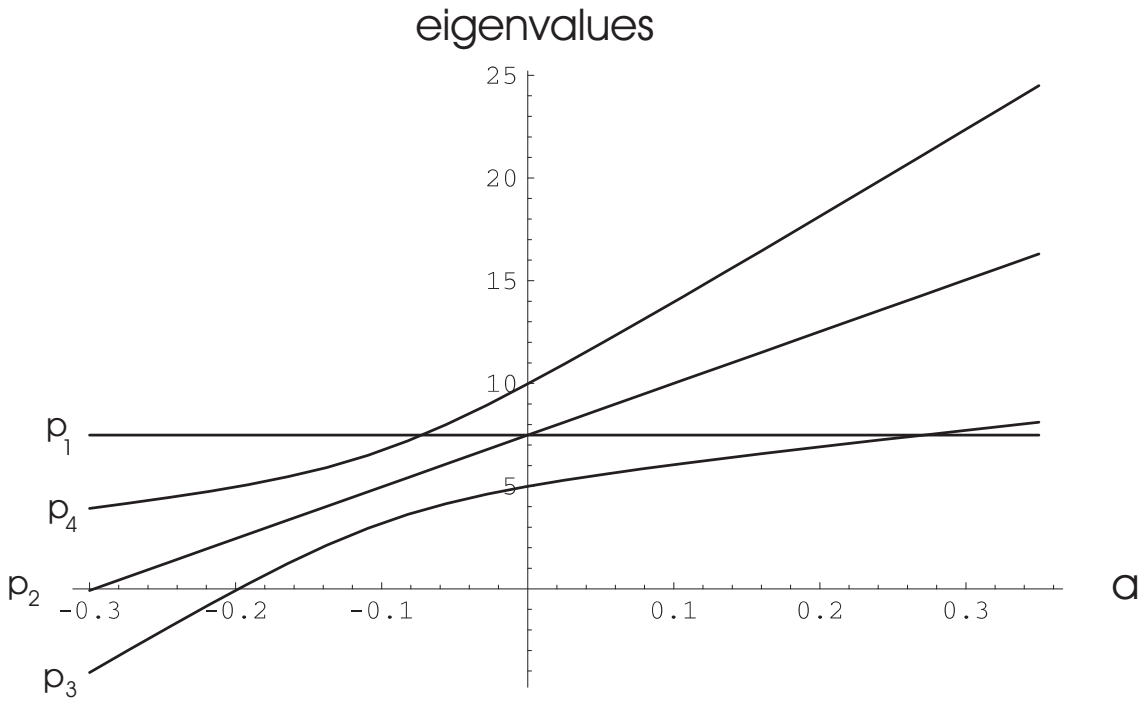

Figure 4.6: *Eigenvalues as a function of the atomic interaction parameter* a. p_1 *does not depend on* a, p_2 *shows a linear,* p_3 *and* p_4 *a hyperbolic dependence on* a. *The plot shows the cases of attractive (*$a < 0$*) and repulsive (*$a > 0$*) interaction between the atoms. The parameters are* $\kappa = 1, \eta = 200, \Delta_c = 10, U_0 = -5$ *and* $\delta = 0.005$.

The corresponding eigenvectors are:

$$v_1 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{4.45}$$

and

$$v_3 = \begin{pmatrix} -\frac{\mathbf{x}}{\sqrt{1+\mathbf{x}^2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{1+\mathbf{x}^2}} \end{pmatrix}, v_4 = \begin{pmatrix} -\frac{\mathbf{y}}{\sqrt{1+\mathbf{y}^2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{1+\mathbf{y}^2}} \end{pmatrix} \tag{4.46}$$

where

$$\mathbf{x} = \frac{(\omega_0 + 2\delta u + av) + \sqrt{(\omega_0 + 2\delta u + av)^2 + a^2 v^2}}{av}, \tag{4.47}$$

$$\mathbf{y} = \frac{(\omega_0 + 2\delta u + av) - \sqrt{(\omega_0 + 2\delta u + av)^2 + a^2 v^2}}{av}. \tag{4.48}$$

The eigenvectors v_1 and v_2 are zero in the first and fourth component and resemble the symmetric and antisymmetric excitation states of the atoms. They are independent of the atomic interaction parameter a.

The eigenvectors v_3 and v_4 are zero in the second and third component and they depend on a such that for $a = 0$ we get the vectors

$$v_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.49}$$

The eigenvectors are printed in Fig 4.5 and 4.5.
Again the plots show the dependence of the eigenvectors for a negative interaction strength $a < 0$, i.e. an attractive interaction between the atoms, and a positive interaction strength $a > 0$, being the characteristic of the scattering by a repulsive interaction. One can interpret the first two eigenvectors as the coefficients c_{12} and c_{21}, which means one atom in the ground state - one atom in the first excited state and vice versa; and the second pair of eigenvectors as the coefficients c_{11} and c_{22}, meaning both atoms in the ground state or both in the first excited state.

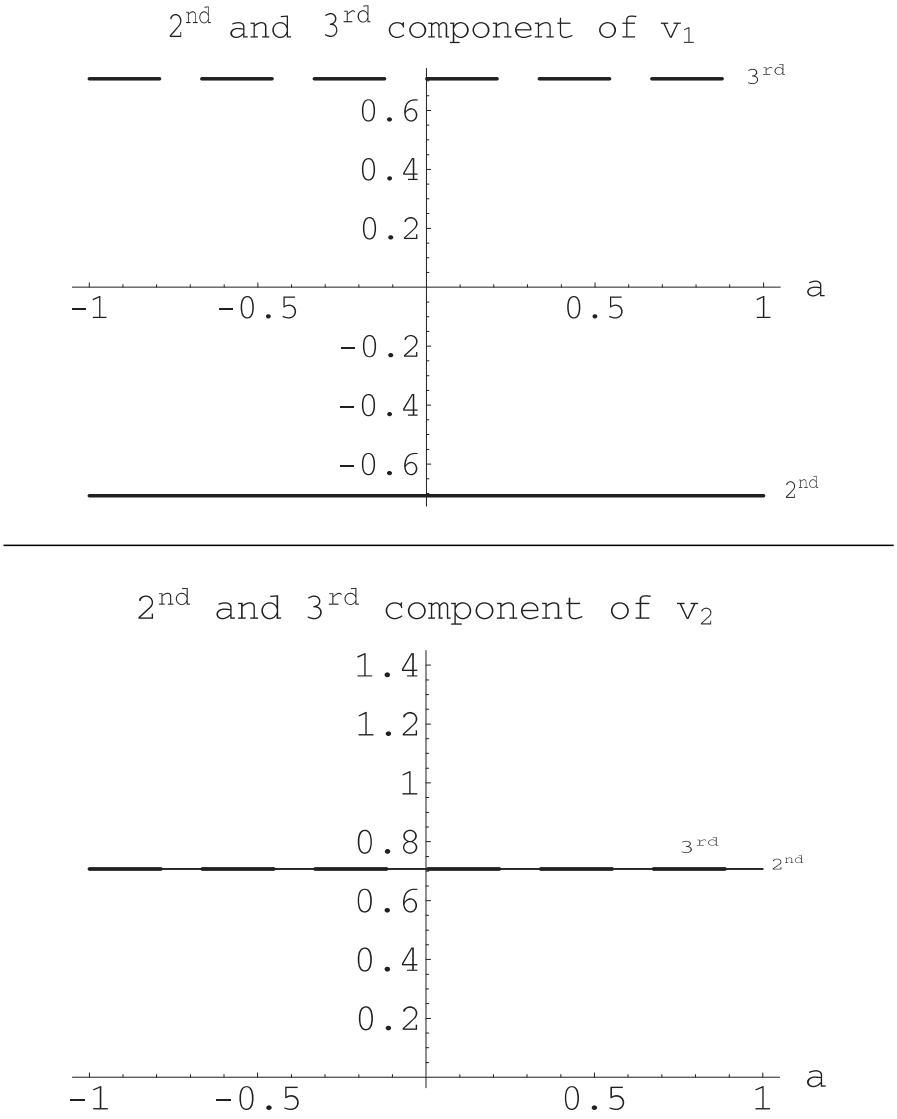

Figure 4.7: *The eigenvectors $v_1(a)$, $v_2(a)$. The eigenvectors v_1 and v_2 are zero in the first and fourth component and independent of the atomic interaction parameter a. The plot shows the cases of attractive ($a < 0$) and repulsive ($a > 0$) interaction between the atoms. The parameters are $\kappa = 1, \eta = 200, \Delta_c = 10, U_0 = -5$ and $\delta = 0.005$.*

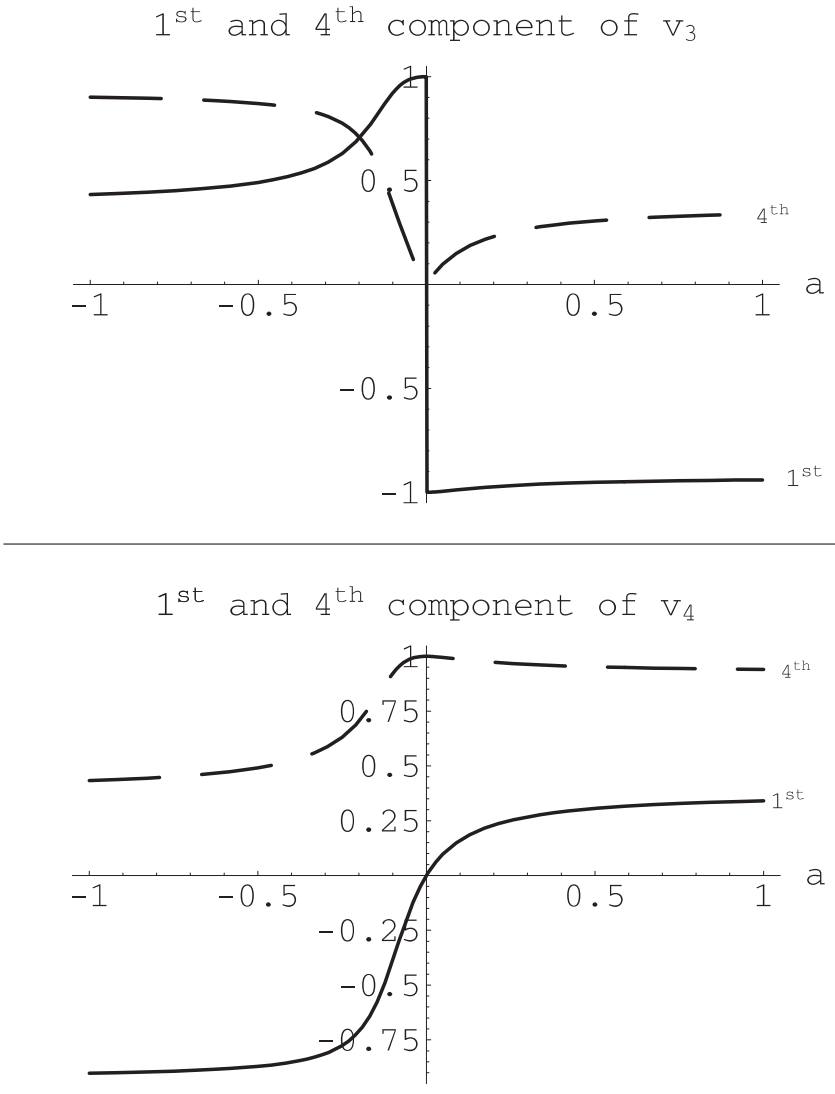

Figure 4.8: *The eigenvectors $v_3(a)$, $v_4(a)$. The eigenvectors v_3 and v_4 are zero in the second and third component and depend on the atomic interaction parameter a. The plot shows the cases of attractive ($a < 0$) and repulsive ($a > 0$) interaction between the atoms. The parameters are $\kappa = 1, \eta = 200, \Delta_c = 10, U_0 = -5$ and $\delta = 0.005$.*

For fixed δ the solution for the coefficients is obtained by transforming the equation for the eigenvalues of the matrix M

$$M\vec{c_i} = \lambda_i \vec{c_i} \tag{4.50}$$

into an equation for the eigenvalues using the eigenvectors of M.

$$D\vec{v_i} = \lambda_i \vec{v_i} \tag{4.51}$$

Where D is the diagonal matrix we find when applying the linear transformation $D = P^{-1}MP$:

$$D = P^{-1}MP = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & p_3 & \\ & & & p_4 \end{pmatrix} , \tag{4.52}$$

where

$$P = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} . \tag{4.53}$$

The solution for equation (4.51) is

$$\vec{v}(t) = e^{-iDt}\vec{v}(0) \tag{4.54}$$

with

$$e^{-iDt} = \begin{pmatrix} e^{-ip_1 t} & & \\ & \ddots & \\ & & e^{-ip_4 t} \end{pmatrix} . \tag{4.55}$$

Doing the reverse transformation, we get for the coefficients

$$\begin{aligned} \vec{c}(t) &= P\vec{v}(t) \\ &= P\, e^{-iDt}\vec{v}(0). \end{aligned} \tag{4.56}$$

And finally, by transforming $\vec{v}(0) = P^{-1}\vec{c}(0)$, we get an equation for the coefficients $\vec{c}(t)$:

$$\vec{c}(t) = P\, e^{-iDt}P^{-1}\vec{c}(0). \tag{4.57}$$

Hence we have derived a solution of Eq. (4.25) in the desired form.

4.6 Stationary state solutions

Our final goal is now to find a general solution to the problem. This motivates to find stationary solutions for each eigenvector v_i and find out if a

general solution by superpositions of the eigenvectors is possible.
We calculate the expectation value for the squared atomic position using the eigenvectors v_i. Hence we get four constant expectation values $< x_i^2 >$:

$$< x_i^2 >= v_i^* A v_i \ , \tag{4.58}$$

with

$$\mathbf{A} = \frac{k^2 x_0^2}{2} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix} \ . \tag{4.59}$$

We now use these solutions $< x_i^2 >$ to calculate the corresponding field amplitudes α_i for each eigenvector v_i:

$$\alpha_i = \frac{\eta}{\kappa - i(\Delta_c - U_0(2 - k^2 < x^2 >_i))} \ . \tag{4.60}$$

Note that $< x_i^2 >$ does not depend on the phase of v_i, so that α_i does not depend on this phase too.

Hence the solutions (4.57) correspond to a constant field and hence constant $\delta(t)$.
The remaining problem is now to find conditions for which δ_i is consistent with α_i. As α_i is a monotonic function of η_i, we can try to find the needed η_i by choosing δ_i suitably.
For this we want to find η as a function of δ_i. We try to find it from the equation for δ_i:

$$\delta_i = U_0(|\alpha_i|^2 - |\alpha_0|^2) \tag{4.61}$$

Note that η also appears in the equation for α_0 and therefore, as we defined $\omega_0^2 = \frac{2}{m} U_0 |\alpha_0|^2$, ω_0 and $x_0 = \sqrt{\frac{\hbar}{m\omega_0}}$ are also functions of η.
The expectation values $< x_i^2 >$ enter the equation for α_i (4.60), hence an analytic calculation of $\eta_i(\delta_i)$ from the above equation (4.61) is only possible for the cases $i = 1$ and $i = 2$, i.e. for the first two eigenvectors v_1 and v_2, whose components do not depend on η_i. If we do so, we get a cubic equation for η_1 and η_2.
For the eigenvectors v_3 and v_4 an analytic treatment is not possible, because the expectation values $< x_3^2 >$ and x_4^2 who enter the equation for the field α_3 and α_4 depend on polynomials in η_3 and η_4 of a non-trivial form.
But if we choose for α_0 a constant $\eta = \eta_0$, ω_0 and x_0 depending on η_0 too,

we can easily calculate $\eta(\delta_i)$ for each eigenvector v_i:

$$\eta_i(\delta_i) = \sqrt{(\frac{\delta_i}{U_0} + |\alpha_0|^2)\Big(\kappa^2 + \big(\Delta_c - U_0(2 - < x_i^2 >)\big)^2\Big)} \tag{4.62}$$

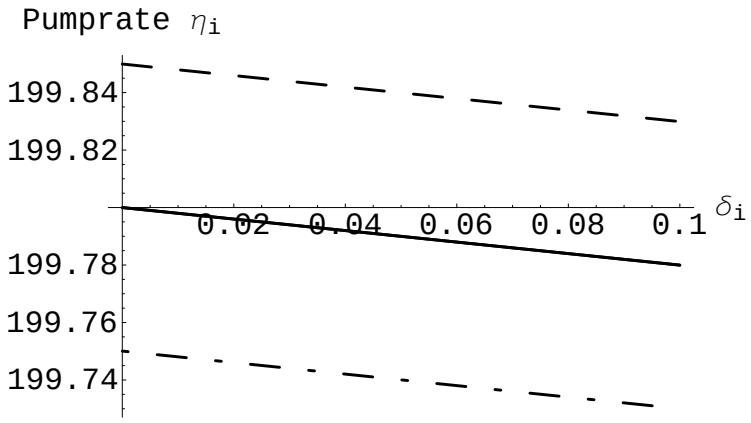

Figure 4.9: *Pump rate η_i in terms of δ_i for each eigenvector v_1, v_2, v_3 and v_4. The plot shows η_1 and η_2 (solid line), which are equal, and η_3 (dashed line) and η_4 (dash-dotted line) being slightly displaced. The interaction parameter is chosen $a = 0,01$ and the operating parameters are $\kappa = 1, \Delta_c = 10, \eta = 200, U_0 = -5$.*

We plot the functions $\eta_i(\delta_i)$, as shown in Fig.4.6, and see that the curves for η_1 and η_2 coincide, as one would expect, since the eigenvectors v_1 and v_2 (Eq.(4.45)) only differ by a minus sign. The curves for η_3 and η_4 are slightly displaced up- and downwards respectively.

A numerical check of the results, where we use the Schrödinger equation (4.25), reveals that the eigenstates with the corresponding η and δ yield stationary solutions.

4.7 Conclusions

We have done a review of the formalism to derive in a rotating frame the Hamiltonian and the master equation for one atom interacting with a single cavity mode coupled to an external reservoir. We also derived the Heisenberg equations of motion for the atomic and field operators under

the assumption of good cavity and low saturation limits. The force acting on the atom was found to push the atom to low intensity regions if the laser field is blue detuned and to high intensity regions for a red detuning respectively.
As a next step we concentrated on the backaction of the atom on the cavity field. We introduced a model for this backaction which resembles the time-dependent frequency as a perturbation $\delta(t)$ to a harmonic potential. We found that the system dynamically traps and localizes itself around an antinode of the field.
The actual field of interest were two interacting atoms in a cavity standing field mode, thus the direct interaction between them and the respective interaction between the atom and field. We applied the model of the perturbed harmonic potential on the case of two atoms in the same field and modelled the direct interaction between the atoms by a delta function parametrized by the interaction parameter a. We found that the atoms periodically exchange energy.
As we are interested in analytic stationary states for the system we calculated the eigenstates of the matrix in the matrix representation of the Schrödinger equation and due to the special shape of the matrix we found two sets of eigenvectors, linearly independent of each other.
We see that there are two possible classes of states. A first class where only one atom is excited, thus the coefficients c_{12} and c_{21} are non-zero, and a second class, where either both or none of the atoms are excited, i.e. the coefficients c_{11} and c_{22} are non-zero.
Dynamics only couples states within its class, that means for example if we have the situation that only one atom is excited, according to our results the situation that both atoms are excited will not occur.
Hence effectively we have two separate two state systems. These states could then be used as a basis of further studies of atoms in periodic lattices [16].
A further interesting research topic would be the extension to three or more states of the atoms or even more than two atoms in the cavity.

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Curriculum vitae

Personal data

Name	Katharina Renz
Date of Birth	November 27, 1977
Place of Birth	Lilienfeld, Austria

Education

1984 - 1988	Elementary school, Innsbruck
1988 - 1996	High school, WRG Ursulinen Innsbruck
June 1996	General qualification for university entrance
since 1996	Diploma studies in physics at the University of Innsbruck