Cavity induced atom cooling and trapping

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Zusammenfassung

Das Kühlen und Einfangen von Atomen in einer quantisierten Stehwelle eines optischen Resonators hoher Güte werden untersucht. Während im ersten Teil ein Überblick über das Grundwissen der semiklassischen Laserkühlung unter besonderer Berücksichtigung des Dopplerkühlens gegeben wird, beschäftigt sich der Rest der Arbeit mit den neuen Effekten, die im Hohlraum auftreten. Es wird ein anschauliches Bild mit Hilfe eines Sisyphus-Kühlschemas für eine nur in verlustarmen Resonatoren vorhandene Kühlkraft gegeben. Diffusion und Temperatur der gefangenen Atome werden abgeleitet. Es wird gezeigt, daß das Dopplerlimit unterschritten werden und eine Temperatur von $k_B T = \hbar\kappa$ erreicht werden kann, wobei κ die Linienbreite des Resonators ist.

Contents

Preface

In my thesis I investigate the interaction of a two level atom with a cavity standing wave in the strong coupling regime. Dissipation through spontaneous emission and the cavity mirrors is included in my model. In the good cavity limit a new cooling mechanism appears which is entirely different from standard Doppler cooling. As a consequence equilibrium temperatures far below the Doppler limit can be reached.

The thesis is organized as follows: In the first chapter I will give a review on the standard theory of semiclassical laser cooling, which is the basis for understanding the new concepts introduced in the following chapters. It will be discussed what is meant by semiclassical approximations and the validity of the semiclassical framework will be investigated. Important physical concepts like that of friction coeffcient and that of diffusion constant will be introduced and the idea of trapping atoms by means of light is presented. Special attention will be paid to the case of an atom moving in a classical standing wave light field. Analytical solutions to the optical Bloch equations and the corresponding force expressions for 'small' velocities are presented. In that context the 'Doppler limit' will be investigated .

The rest of the thesis will be concerned with an atom in a standing wave light mode inside a cavity. It is well known that radiative properties are changed inside a cavity. The mode structure in a cavity is different from free space, which led Pourcel to predict enhancement in the spontaneous decay rate for an atom placed in a resonant cavity [1]. Kleppner, on the other hand, pointed out the possibility of inhibiting spontaneous emission [2]. Also the Jaynes Cummings model [3] in the 'good cavity limit' has been investigated in detail. A scheme to cool free atoms in colored vacua characteristic for cavities was proposed in recent papers [4, 5]. The situation changes again in the case of an external coherent light source driving the atom inside the cavity and effects like positive stationary population inversion [6] and changes in the atomic decay rates [7] are some of the possible consequences. Another paper dealt with the interaction between the cavity mode and an atom , which was additionally strongly driven by a laser, in the bad cavity limit [8]. In experiment, however, photons must be fed into the cavity and photons can also leak out of the cavity. No proposals for cooling an atom in a driven cavity have been made so far. Similarly to the driven atom

case, one also expects new physical effects to appear. A necessary prerequisite for that is the strong influence of a single atom on the cavity dynamics. This can only be reached if the atom-mode coupling g is stronger than cavity decay κ and spontaneous emission Γ ('good cavity' limit), which was only made possible by the advent of better and better optical cavities in the past years. In this limit Turchette, Kimble et al. [9, 10] demonstrated experimentally the strong influence of a single atom on transmission and intracavity intensity of a weakly driven cavity. This is also the regime we are interested in, as due to strong coupling and small decay rates only small numbers of atoms and photons are necessary to significantly affect the system as a whole [9]. It is based on those experimental achievements that the cooling scheme proposed in this work becomes feasible.

Therefore the model of an atom interacting with a quantised standing wave in a weakly driven cavity will be presented. In this model the intracavity intensity is strongly dependent on the atomic position. An expression for the friction force is analytically derived. Subsequent adiabatic elimination of cavity mode or atomic operators respectively shows that the total friction force can be seen as the sum of two forces. The first contribution is shown to be related to standard Doppler cooling whereas the second contribution arises from the cavity dynamics and is a new kind of cooling force ('cavity cooling'). Driving the cavity near resonance it turns out to be the dominant force and a simple physical picture of it in terms of dressed states and Sisyphus cooling [11] is given. A derivation of the the diffusion using the Quantum Regression Theorem (QRT) shows that an additional term also arises in the diffusion, which does not appear in a classical treatment of the cavity standing wave. Again an interpretation of this term is given in terms of fluctuations in the Sisyphus forces. It is shown that equilibrium temperatures in the order of $\hbar\kappa$ can be achieved, which can be significantly below the Doppler limit for sufficiently 'good' cavities. Finally, we make comparisons to results obtained through a fully quantum treatment of the problem including quantization of the atomic motional degrees of freedom which are in very good agreement to the semiclassical results.

Chapter 1

General theory of laser cooling

Einstein already pointed out in his fundamental work on radiative absorption and emission processes of an atom in a thermal field that the momentum exchange between light and matter could be used to slow atomic motion [12]. Essentially the atom will come into thermic equilibirum with its surroundings. When the surrounding electromagnetic field has a spectral distribution corresponding to $0K$ (monochromatic light) the atom would also be expected to approach this temperature and would so transfer its kinetic energy to the surrounding heat bath (The atomic temperature is actually limited by the width of the transfer channel, i.e. the atomic linewidth). In fact, experiments of that kind have recently been done with diffuse light [13].

One can get a better understanding of the underlying physical mechanisms by considering the interaction of an atom with a plane travelling or standing electromagnetic wave, described by a c-number field:

$$\mathbf{E}(x,t) = \vec{\epsilon} A(x) \frac{1}{2} e^{-i\omega t - i\phi(x)} + c.c., \tag{1.1}$$

where $A(x)$ is the space dependent amplitude, $\phi(\mathbf{x})$ the space dependent phase and $\vec{\epsilon}$ the polarization of the laser electric field. In a quantum picture it would be described by a similar expression containing field operators:

$$\mathbf{E}_L(x,t) = i\vec{\epsilon} u(x) \left(a e^{-i\omega_P t - i\phi(x)} - a^+ e^{+i\omega_P t + i\phi(x)} \right). \tag{1.2}$$

Their expectation values evaluated for a coherent field would yield the same. The interaction of the laser with the atom in the standard dipole approximation (long wavelength approximation), where the interaction between the atom and the electromagnetic field is only expanded to first order in $\frac{x}{r}$, can be characterized by the following Hamiltonian:

$$H = H_A + H_V + V_{AL} + V_{AV}, \tag{1.3}$$

where

$$H_A = \frac{P^2}{2m} + \hbar\omega_{10}|e\rangle\langle e| \tag{1.4}$$

is the atomic Hamiltonian for the external and internal degrees of freedom respectively.

$$H_V = \sum_k \hbar\omega_k \left(a_k^+ a_k + \frac{1}{2}\right) \tag{1.5}$$

is the Hamiltonian describing the vacuum free field (heat bath).

$$V_{AL} = -\mathbf{d}\mathbf{E}_L(x,t) \qquad V_{AV} = -\mathbf{d}\mathbf{E}_V(x) \tag{1.6}$$

are the interaction terms between atomic dipole $\hat{d}$ and the laser and vacuum electric field respectively. The vacuum electric field can be expanded as follows

$$\mathbf{E}_V(x) = i\sum_k \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\vec{\epsilon_k} a_k e^{ik\mathbf{x}} + h.c., \tag{1.7}$$

where V is the volume of quantization. Now we also make the rotating wave approximation, which keeps only energy conserving terms and which is valid for quasi resonant interaction (like in the case we consider). Expressing the dipole operator

$$\mathbf{d} = \vec{d}\left(\sigma^+ + \sigma^-\right) \tag{1.8}$$

and inserting it into (1.6) applying the rotating wave approximation yields

$$V_{AL} = \frac{\hbar\omega_R(\mathbf{x})}{2}\left(e^{-i\omega_P t - i\phi(x)}\sigma^+ + h.c.\right), \tag{1.9}$$

where

$$\omega_R(x) = \vec{\epsilon}\vec{d}A(x) \tag{1.10}$$

is the associated Rabi frequency. In the same way one only retains the terms $\sigma^+ a, a^+\sigma^-$ in V_{AV} and omits the nonresonant terms.

For the force acting on the atom we find [14, 15]:

$$\begin{aligned} \mathbf{F} &= \dot{\mathbf{P}} \\ &= \frac{i}{\hbar}[H, \mathbf{P}] \\ &= -\nabla V_{AV}(x) - \nabla V_{AL}(x). \end{aligned} \tag{1.11}$$

As only these contributions to the total Hamiltonian depend on x, the gradient acts only on the atom-laser and the atom-vacuum coupling with their respective space dependencies. Forming the Heisenberg equations for the field operators we find

$$\dot{a}_k = -i\omega_k a_k - \sqrt{\frac{\omega_k}{2\epsilon_0 V\hbar}} e^{-ikx}\sigma^- \tag{1.12}$$

and upon integration

$$a_k(t) = a_k(0)e^{-i\omega_k t} + \int_0^t dt' e^{-i\omega_k(t-t')}\sqrt{\frac{\omega_k}{2\epsilon_0 V\hbar}} e^{-ikx}\sigma^-(t'). \tag{1.13}$$

The field operators can be split into a freely evolving term and into a source term radiated by the atomic dipole. Inserting this into $\mathbf{E}_V$ one obtains

$$E_V(\tilde{x},t) = \mathbf{E}_{free}(\tilde{x},t) + \mathbf{E}_{source}.(\tilde{x},t) \tag{1.14}$$

The source term is found by summing up all the contributions of the individual source components over the set of wavenumbers k. The components depend on k through the frequency ω_k, and the exponential $e^{ik(x-\tilde{x})}$, which upon summation(integration) over k yields an even function of $x-\tilde{x}$, the gradient of which vanishes when evaluated at the position of the atomic centre of mass x. This means that the electric dipole field does not exert force on the dipole itself, which is well known from classical electrodynamics. The remaining free evolving term only contains field operators $a(0), a^+(0)$ linearly, with the consequence

$$\langle -\nabla \mathbf{E}_{free}(\mathbf{x},t)\rangle = 0$$

when averaged over the vacuum field.

Note that the commutator of $\mathbf{E}^-_{free}$ with $\mathbf{E}^+_{free}$ does not average to zero, which gives rise to fluctuations in the force and thus to diffusion (see below). As for the moment we are interested in the mean force only, we are left with the contribution due to the atom-laser coupling $-\nabla V_{AL}$.

Up to now we have assumed $\hat{x}$ to be the position operator for the atomic centre of mass. Assuming well localized wave packets for the atom one can replace its expectation value by the classical coordinates of the atomic centre of mass and neglect the quantum mechanical nature of the external degrees of freedom [16, 14, 17].

Later I will give a brief account on the necessary requirements for this so called 'semi classical' approximation. Now one can change into a frame rotating with the laser frequency ω_P and get rid of the explicit time dependence in V_{AL}. One obtains for the force operator:

$$\mathbf{F}(x) = -\frac{\hbar\nabla\omega_R(x)}{2}\left(\sigma^+ e^{-i\phi(x)} + \sigma^- e^{+i\phi(x)}\right) - i\nabla\phi(x)\frac{\hbar\omega_R(x)}{2}\left(\sigma^+ e^{-i\phi(x)} - \sigma^- e^{+i\phi(x)}\right). \quad (1.15)$$

Its mean value can be obtained by tracing over the internal degrees of freedom.:

$$f(x) = Tr\{\rho F(x)\} \quad (1.16)$$
$$= -\hbar\nabla\omega_R(x)u_x(t) - \hbar\omega_R\nabla\phi(x)v_y(t), \quad (1.17)$$

where $u_x(t)$ and $v_y(t)$ are the components of the 'Bloch vector' [1]

$$u_x(t) = \mathcal{R}e\left\{\rho_{ge}(t)e^{-i\phi(x)}\right\} \quad (1.18)$$
$$v_y(t) = \mathcal{I}m\left\{\rho_{ge}(t)e^{-i\phi(x)}\right\}. \quad (1.19)$$

This yields the dipole force, which is proportional to the intensity gradient $\nabla\omega_R(x)$ of the laser and the radiation pressure force proportional to the phase gradient $\nabla\phi(x)$ of the laser. What is left to do is to calculate the density matrix elements and insert them into the expression for the force.

The parts of the total Hamiltonian describing the vacuum free field H_V and the coupling of the atomic dipole to the vacuum modes V_{AV} can be replaced by a Liouvillian operator in a density matrix equation for the atomic operators alone [18]. This enables us to restrict ourselves to the internal atomic degrees of freedom while we get rid of the inifinte set of vacuum modes the dissipative action of which is accounted for by the Liouvillian in the atomic density matrix equation, which is presented in a rotating frame below. One changes into the rotating interaction picture by the unitary transformation[2]

$$U(t) = e^{-i\omega_P t(\sigma^+\sigma^-)} \quad (1.20)$$

and thus eliminates the time dependence in the in the interaction term as well as introducing the frequency difference

$$\Delta_a = \omega_P - \omega_{10}$$

in the atomic Hamiltonian. The standard master equation is then found to be

[1] They are the σ^x, σ^y expectation values for $\phi(x) = 0$

[2] Note that $\mathbf{F}$(x) of (1.15) has already been transformed into the rotating frame

$$\dot{\rho} = -\frac{i}{\hbar}\left[H,\rho\right] - \Gamma\left(\left\{\sigma^+\sigma^-,\rho\right\}_+ - 2\sigma^-\rho\sigma^+\right), \tag{1.21}$$

where

$$H = -\hbar\Delta_a\sigma^+\sigma^- + \frac{\hbar\omega_R(x)}{2}\left(\sigma^+ + \sigma^-\right). \tag{1.22}$$

Γ is the spontaneous decay rate of the excited atomic state (HWHM at resonance)[3].

If one expresses the master equation above in terms of excited and ground states of the atom ($|e\rangle, |g\rangle$), one obtains the following equations:

$$\begin{aligned}
\dot{\rho}_{ee} &= i\frac{\omega_R}{2}\left(\rho_{eg}e^{+i\phi(x)} - \rho_{ge}e^{-i\phi(x)}\right) - 2\Gamma\rho_{ee} \\
\dot{\rho}_{gg} &= -i\frac{\omega_R}{2}\left(\rho_{eg}e^{+i\phi(x)} - \rho_{ge}e^{-i\phi(x)}\right) + 2\Gamma\rho_{ee} \\
\dot{\rho}_{ge} &= -i\Delta_a\rho_{ge}e^{-i\phi(x)} - i\frac{\omega_R}{2}\left(\rho_{ee} - \rho_{gg}\right) - \Gamma\rho_{ge}e^{-i\phi(x)} \\
\dot{\rho}_{eg} &= \dot{\rho}^*_{ge}.
\end{aligned} \tag{1.23}$$

One of them is redundant as the populations sum up to one:

$$\rho_{ee} + \rho_{gg} = 1.$$

It is more convenient to introduce the quantities $u_x(t)$ and $v_y(t)$ defined above and an additional quantity $w_z(t) = \frac{1}{2}(\rho_{ee} - \rho_{gg})$, which can be related to the expectation values of the cartesian components of a pseudospin 1/2 [19]. This set of equations is called Bloch equations. The density matrix equations (1.23) can be solved for the case of a motionless atom in steady state case ($\dot{u} = 0, \dot{v} = 0...$) to yield

$$\begin{aligned}
u_x &= \frac{\Delta_a}{\omega_R}\frac{s}{1+s} \\
v_y &= = \frac{\Gamma}{\omega_R}\frac{s}{1+s} \\
\rho_{ee} &= \frac{1}{2} + w_z \\
&= \frac{1}{2}\frac{s}{1+s},
\end{aligned} \tag{1.24}$$

where s is the saturation parameter defined as

$$s = \frac{\omega_R^2/2}{\Delta_a^2 + \Gamma^2}. \tag{1.25}$$

[3]half width at half maximum

1.1 Plane wave : the radiation pressure

For the case of a plane wave one has $\phi(x) = -kx$, where k is the wavevector, and the intensity along the wave is constant. Thus

$$\nabla\omega_R(x) = 0 \tag{1.26}$$
$$\nabla\phi(x) = -k. \tag{1.27}$$

So one finds for the radiation pressure force

$$\begin{aligned} f_{rp}(x) &= \hbar\omega_R(x)kv_y \\ &= \hbar k\Gamma\frac{\omega_R^2/2}{\Delta_a^2+\Gamma^2+\omega_R^2/2}. \end{aligned} \tag{1.28}$$

This expression shows that the radiation pressure is a dissipative force as it varies like an absorption curve with Δ_a. For high saturation it reaches a maximum of

$$f_{max} = \hbar k\Gamma$$

corresponding to accelerations in the range of $10^5 g$ for Na-atoms. For an atom moving in a plane wave with constant velocity, the Bloch equations can again be easily solved analytically for the steady state. The atom sees the incoming photon frequency shifted by

$$\tilde{\omega}_P = \omega_P - \mathbf{kv} \tag{1.29}$$

and thus $\Delta_a = \omega_P - \omega_{10}$ in (1.28) must be substituted by $\Delta_a - \mathbf{kv}$ to obtain the force for the moving atom. The resulting velocity dependent force can be expanded in powers of v to obtain the friction coefficient β as defined

$$f_\beta = f(v=0) + \beta v + O(v^2). \tag{1.30}$$

One finds

$$\beta = \hbar k^2\omega_R^2\frac{\Delta_a\Gamma}{[\Delta_a^2+\Gamma^2]^2}. \tag{1.31}$$

The radiation pressure force (1.28) can be understood as the product of the transferred photon momentum $\hbar k$ and the photon scattering rate, which is the rate of absorption-spontaneous emission cycles. The average momentum transfer of one cycle equals $\hbar k$ due to the symmetrical distribution of spontaneous emission. So the radiation pressure force can also be written [4] as $\hbar k 2\Gamma\rho_{ee}$.

It can be used in atomic beam deflection, deceleration and velocity collimation experiments [20].

[4] As stimulated emission goes into the beam direction, absorption- stimulated emission cycles do not contribute to the force

1.2 Standing wave : the dipole force

For a laser standing wave the intensity varies as

$$\omega_R(x) = \omega_0 cos(x)$$

along the cavity axis while the the phase $\phi(x)$ remains constant. This is also what we will be interested in when discussing the interaction of the atom placed inside a weakly driven cavity, where the atom interacts with the cavity standing wave. For the case of the motionless atom the Bloch equations yield the steady state dipole force, which can be immediately found by substituting eqn.(1.24) into eqn.(1.17):

$$\begin{aligned} f_{dp}(x) &= -\nabla\omega_R(x)u_x \\ &= -\nabla\omega_R(x)\frac{\Delta_a}{\omega_R}\frac{s}{1+s} \\ &= -\frac{\hbar\Delta_a}{4}\frac{\nabla\omega_R^2}{\Delta_a^2 + (\Gamma)^2 + \omega_R^2/2}. \end{aligned} \tag{1.32}$$

One sees, that $f_{dp}(x)$ varies like a dispersion curve [5] with respect to Δ_a. The dipole force reaches a maximum for ω_R in the order of Δ_a:

$$f_{dp}^{max} \approx \hbar k \omega_R.$$

As a standing wave is the sum of two opposite running waves the dipole force can be associated with absorption of photons from one component of the standing wave and stimulated emission into the other component. The net momentum transferred to the field also changes the atomic momentum. It can be shown that energy is only absorbed through the dissipative radiation pressure but no energy is absorbed by the dipole force, where photons of the same frequency are absorbed from and reemitted into the standing wave.

The expression of the dipole force shows that atoms are attracted to high intensity regions for $\Delta_a < 0$ and it is said that the atom is a 'high field seeker'. The dipole force can be related to an optical potential

$$V(x) = \frac{\hbar\Delta_a}{2} ln\left[1 + \frac{\omega_R^2(x)}{\Delta_a^2 + \Gamma^2}\right]. \tag{1.33}$$

For an atom moving with small velocity inside the standing wave the Bloch equations cannot be solved analytically as the coefficients of the linear differential equations depend sinusoidally on x through $\omega_R(x)$. However, numerical results were obtained by the method of continued fractions [21, 22], where

[5] A two level system including dissipative processes can be seen completely analogous to the the Rabi oscillations of an electron subjected to a sinusoidally varying magnetic field which are damped by an additional decay channel. There the σ_x and σ_y expectation values also show absortpive and dispersive character respectively.

Fourier expansions of the force were calculated. We want to find solutions only to first order in $\frac{kv}{\Gamma}$ and therefore follow a perturbative treatment first suggested by Gordon and Ashkin [23], which yields the same result as computing the first few spatial components of the force. One can expand the density matrix into a zeroth and first order term in v. Retaining only components to first order in v and noting that

$$\dot{\rho} = \left\{ \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \right\} \rho,$$

where $\frac{\partial}{\partial t}$ can be omitted as we are looking for the steady state solutions, we obtain

$$v\frac{\partial}{\partial x}\rho_0 = \mathcal{L}\rho_1, \tag{1.34}$$

where

$$\mathcal{L}\rho = -\frac{i}{\hbar}\left[H, \rho\right] - \Gamma\left(\sigma^+\sigma^-\rho + \rho\sigma^+\sigma^- - 2\sigma^-\rho\sigma^+\right). \tag{1.35}$$

This means that we substitue the zeroth order solution on the left side to obtain a matrix equation for the first order solution on the right side. We will concentrate only on the weak saturation limit ($s \ll 1$) which will be needed later when extending our model. Inverting the above matrix equation[6], substitutung the solution found for u_x to first order in the velocity into (1.17) we find

$$f_\beta = \hbar k^2 \left(\frac{\omega_0}{2}\right)^2 \frac{2\Delta_a\Gamma}{\left[\Delta_a^2 + \Gamma^2\right]^2} 2sin^2(kx)v. \tag{1.36}$$

Averaging over a wavelength this yields

$$\beta_{dp} = \hbar k^2 \left(\frac{\omega_0}{2}\right)^2 \frac{2\Delta_a\Gamma}{\left[\Delta_a^2 + \Gamma^2\right]^2}. \tag{1.37}$$

Comparing it with equation (1.31) for the friction coefficient of a plane wave one sees that it is the sum of the friction coefficient of the wave propagating to the left and the wave propagating to the right[7]. The interpretation of this cooling mechanism, called 'Doppler cooling' [24, 25], is that owing to the Doppler effect, the moving particle tends to absorb photons from the laser wave counterpropagating its velocity rather than from the copropagating wave, if the laser is tuned below resonance.

[6] after expressing ρ in terms of $|e\rangle, |g\rangle$,given by eqns.(1.23)

[7] The amplitude ω_0 of a standing wave must be associated with two times the amplitude of the running waves of which it is comprised to yield the same averaged intensity., $\omega_R^{Run.wave} \rightarrow \omega_0/2$ in this comparison.

One has to bear in mind, however, that this applies only to the low intensity limit, where interference effects can be neglected. Note further that the friction force becomes zero at the antinodes, the region where atoms are supposed to collect as they are 'high field seekers'. This has implications for the attempt to trap them stably in a potential pot located at the antinode. For strong intensities a different approach over dressed states may be taken [11].

1.3 The semiclassical approximation

One question remaining to be answered is under which circumstances the replacement of the $\hat{x}$ operator by a classical variable is valid [17]. For this to hold, one certainly has to assume, that the spatial spread of the atomic wavepacket is much smaller than the laser wavelength [16].

$$\Delta X k \ll 1 \tag{1.38}$$

Furthermore, the momentum spread of the wavepacket must induce only small Doppler shifts compared to the natural linewidth:

$$\frac{k\Delta P}{m} \ll \Gamma. \tag{1.39}$$

Considering the Heisenberg inequality

$$\Delta X \Delta P \geq \frac{\hbar}{2}$$

one then finds

$$E_{rec} = \frac{\hbar k^2}{2m} \ll \hbar\Gamma. \tag{1.40}$$

Thus the atomic recoil energy must be much smaller than the linewidth as a necessary condition for a semiclassical treatment. This ensures that the atom will still be on resonance after emitting several photons. Considering that external processes have a timescale in the order of $T_{ext} = \hbar/E_{rec}$ and internal processes a timescale in the order of $T_{int} = \hbar/\Gamma$ one finds

$$T_{ext} \gg T_{int}. \tag{1.41}$$

So on sees that a clear separation between external and internal time scales is a necessary precondition for the application of semiclassics. This allows to regard the velocity of the atom (varies on a scale of T_{ext}) as constant while averaging over the fast changing internal variables.

1.4 Diffusion and the Doppler limit

Although there is a force slowing down and cooling the atoms if the detuning is correctly chosen, there is also a source of heating counteracting this process. So far we have only calculated time averaged mean forces, but the force itself arises from discrete photon momentum transfers and fluctuates around its mean value, which increases the momentum variance. This gives rise to diffusion D [23, 14]. It is defined as

$$2D = \frac{d}{dt}\Delta P^2(t). \tag{1.42}$$

There are several distinct components of diffusion with different physical orgins. For both cases of a standing and a travelling wave there is diffusion due to the random directions of the spontaneously emitted photons. Considering that the atomic momentum undergoes a random walk in momentum space, the corresponding change of the variance in time should be determined by[8]

$$\begin{aligned} 2D_{SE} &= \hbar^2 k^2 2\Gamma \rho_{ee} && (1.43)\\ &= \hbar^2 k^2 \Gamma \frac{s}{s+1}. && (1.44) \end{aligned}$$

For a travelling wave another source of diffusion is the randomness in the number of absorbed photons. Similarly one can deduce

$$D_{abs} = D_{SE}.$$

Things change for a standing wave, where a different contribution D_{dp} to the diffusion due to fluctuations in the dipole force, arises. Similarly to the dipole force, which in the weak intensity limit can be understood from the contributions of two counterpropagating travelling waves, also D_{dp} can be related to D_{abs}. Whereas for the radiation pressure in a traveling wave the randomness results from the random number of photons being absorbed, for the dipole force it can be thought of resulting from the random direction of absorption of the incoming photons. So in the **weak intensity limit** D_{dp} can be set equal to $2D_{abs}$, arising from the contributions of the two running waves. Note however, that like when thinking of the standing wave friction coefficient as arising from the friction coeffiecients of two travelling waves, the space dependence has been averaged out in the expression for D_{dp}. A more accurate analysis for the weak intensity limit yields [23]

[8] This rough estimate coincides with the result obtained through a more thorough calculation. For a more serious discussion of the diffusion occuring in our 'driven cavity model' see below.

$$\begin{aligned} D &= D_{SE} + D_{dp} \\ &= \hbar k^2 \frac{s(\omega_0)}{2} \Gamma (sin^2(kx) + cos^2(kx)), \end{aligned} \tag{1.45}$$

so that the overall space dependence cancels out again. $s(w_0)$ stands for the saturation parameter evaluated on the standing wave amplitude ω_0.

$$s = \frac{\omega_0^2/2}{\Gamma^2 + \Delta_a^2}$$

Now the question remains to which temperatures atoms may be cooled when cooling and heating through diffusion are taken into account [26, 27, 17]. Let us give a simple guess. We know from above that for small velocities

$$f = -\beta * v$$

and so

$$\begin{aligned} \frac{dP}{dt} &= -\frac{\beta}{m} P \\ \frac{dP^2}{dt} &= -\frac{2\beta}{m} P^2, \end{aligned}$$

and from the definition of the diffusion

$$\frac{dP^2}{dt} = 2D.$$

Both cancel in steady state so that

$$\frac{\beta}{m} p^2 = D.$$

Accordingly we obtain for the temperature in our one dimensional model

$$\frac{k_B T}{2} = \frac{P^2}{2m} = \frac{D}{2\beta}. \tag{1.46}$$

So from the expression(1.37) for the friction coefficient evaluated at $\Delta_a \approx -\Gamma$ for maximum damping and the expression(1.45) for the diffusion at low saturation one finds for the minimum temperature attainable

$$k_B T \approx \hbar \Gamma, \tag{1.47}$$

which is called the Doppler limit. It has also been investigated more rigorously in treatments including the quantization of the motional degrees of freedom [28].

Chapter 2

Mechanical light effects in a driven cavity

In the preceding chapter the laser mode the atom was interacting with was assumed to be in a quasi coherent state so that replacement of the mode operators by c-numbers was possible. Standing waves formed by two counterpropagating laser beam would very well apply to that situation. A standing wave also forms in a cavity. However, things are different for that case. In our model, like in real experiment, there is a coupling introduced between the cavity mode and a laser beam driving the cavity via the cavity mirrors. There is also a coupling between the cavity mode and the vacuum field which accounts for the depletion of the cavity mode (cavity decay). In other words, photons are fed into the cavity and may leak out of it. A schematic representation of the system is given in fig. 2.

For strong coherent pumping many photons are in the cavity and one expects the cavity to be in a coherent state with constant intensity. Treating the mode classically as a c-number is then justified. However, when the number of intracavity photons is small and the pumping is weak, the 'nonclassical behavior' of the cavity mode is likely to introduce new cooling effects and diffusion terms, which cannot be derived from the classical model above. For this we quantise the cavity mode and add the following terms to the Hamiltonian (1.3) above:

$$H_C = \hbar\omega_c a_c^+ a_c \tag{2.1}$$

$$V_{CV} = \int_{\Delta\omega} d\omega g(\omega) \left(a_c a_\omega^+ + a_\omega a_c^+\right) \tag{2.2}$$

$$V_{AC} = \hbar g \left(a^+\sigma^- + \sigma^+ a\right) \tag{2.3}$$

$$H_P = -i\hbar\eta \left(a e^{+i\omega_P t} - a^+ e^{+i\omega_P t}\right), \tag{2.4}$$

where H_c is the free cavity Hamiltonian and V_{CV} is the cavity vacuum interaction Hamiltonian. Here the sum over the vacuum modes has been converted

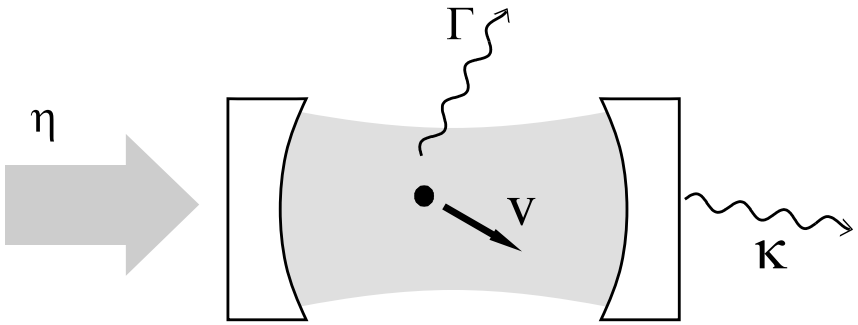

Figure 2.1: Particle moving in a weakly driven cavity with losses from spontaneous emission Γ and cavit decay κ.

into a frequency integral and like in the case of the atom interacting with the vacuum field, the interaction is restricted to a certain bandwidth $\Delta\omega$ [18]. Also the rotating wave approximation was made and the coupling $g(\omega)$ is assumed to be quasi constant over the range of integration. Note that the coupling $g(\omega)$ is not space dependent, as the vacuum field couples to the mode only at the mirror. H_P is the term describing the coherent pumping of the cavity from a coherent laser over the cavity mirror. Note that the term V_{AL} of eqn.(1.9) is written now in the context of a quantized cavity standing wave (ϕ=0) interacting with the atom as

$$V_{AC} = \frac{\hbar\omega_R(x)}{2}\left(a\sigma^+ + \sigma a^+\right),$$

where

$$\begin{aligned}\frac{\hbar\omega_R(x)}{2} &= \hbar g(x) \\ &= \hbar\sqrt{\frac{\omega_c}{2\epsilon_0 V\hbar}}\vec{\epsilon}\vec{d}cos(kx). \end{aligned} \tag{2.5}$$

Note that the coupling strength $\omega_0/2 = g_0$ depends on the quantization Volume V and becomes larger for smaller cavities with smaller volume V. This is important in creating large coupling between cavity and atom in comparison to the cavity decay κ and spontaneous emission Γ (good cavity limit), which has been achieved in the past years with the advent of better optical cavities. In this conetxt a paramter N_0 , the critical atom number, may be introduced which decribes the number of atoms necessary to affect significantly the system as a whole [9, 10]:

$$N_0 = \frac{\kappa\Gamma}{2g_0^2}. \tag{2.6}$$

Now one can eliminate the vacuum field degrees of freedom again and derive a master equation for the atom-cavity mode density operator alone. To get rid of the time dependence appearing in H_P one changes to a rotating (interaction) frame by the unitary transformation

$$U(t) = e^{-i\omega_P t\left(\sigma^+\sigma + a_c^+ a_c\right)}$$

introducing the frequency shifts in the transformed Hamiltonian

$$\Delta_a = \omega_P - \omega_{10} \qquad \Delta_c = \omega_P - \omega_c. \tag{2.7}$$

Finally one finds for the master equation (in a rotating frame) for a two level atom inside a laser cavity interacting with a cavity standing wave while the cavity is driven by a coherent source with losses through spontaneous emission and cavity damping included:

$$\begin{aligned}\dot{\rho} = -\frac{i}{\hbar}\left[H_{J.C.},\rho\right] - \frac{i}{\hbar}\left[H_P,\rho\right] - \Gamma\left(\left\{\sigma^+\sigma^-,\rho\right\}_+ - 2\sigma^-\rho\sigma^+\right) \\ -\kappa\left(\left\{a^+a,\rho\right\}_+ - 2a\rho a^+\right),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}H_{J.C.} = -\hbar\Delta_a\sigma^+\sigma^- - \hbar\Delta_c a^+a + \hbar g\left(a\sigma^+ + \sigma^- a^+\right) \\ H_P = -i\hbar\eta\left(a - a^+\right)\end{aligned} \tag{2.9}$$

are the expressions for the Jaynes Cummings and the Pump Hamiltonian respectively.

2.1 Heisenberg equations

An equivalent way of treating this problem is via the Heisenberg equations for the atomic and mode operators. One finds

$$\begin{aligned}\dot{a} &= i\Delta_c a - ig(x)\sigma^- - \kappa a + \eta + F_1 \\ \dot{\sigma^-} &= i\Delta_a\sigma^- + ig(x)\sigma_z a - \Gamma\sigma^- + \sigma_z F_2,\end{aligned} \tag{2.10}$$

where F_1 and F_2 are noise operators which essentially contain only free input field operators. It is important to know that their expectation values are zero when evaluated for a heat bath at $T = 0$:

$$F_1|vac\rangle = 0 = F_2|vac\rangle. \tag{2.11}$$

Now we want to linearise the above equations. For the averages one has to note that

$$\sigma_z a = (|e\rangle\langle e| - |g\rangle\langle g|) \left(|0\rangle\langle 1| + \sum_{n=2} \sqrt{n}|n-1\rangle\langle n| \right).$$

If the cavity is very weakly driven there is at the most one photon in the cavity and the atom is in the ground state or there is no photon at all and the atom is in the ground or the excited state. In that case states $|e,1\rangle, |g,2\rangle$ or states of even higher photon number do not contribute and can be omitted. Thus

$$\langle \sigma_z a \rangle = -\langle a \rangle.$$

But even for a higher number of photons in the cavity this relation holds approximately, if the population of the ground state is much larger than the population of the excited state, i.e. the saturation of the transition is very small. In that case and considering (2.11) the equations for the expectation values may be written as

$$\langle \dot{\vec{Y}} \rangle = \mathbf{A} \langle \vec{Y} \rangle + \vec{Z}_\eta, \tag{2.12}$$

where

$$\vec{Y} = \begin{pmatrix} a \\ \sigma^- \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} i\Delta_c - \kappa & -ig(x) \\ -ig(x) & i\Delta_a - \Gamma \end{pmatrix} \quad \vec{Z}_\eta = \begin{pmatrix} \eta \\ 0 \end{pmatrix}. \tag{2.13}$$

These equations can be solved for $\langle \dot{a} \rangle = 0 = \langle \dot{\sigma^-} \rangle$ to obtain

$$\langle a \rangle_0 = \eta \frac{\Gamma - i\Delta_a}{det(A)} \tag{2.14}$$

$$\langle \sigma^- \rangle_0 = \eta \frac{-ig}{det(A)}, \tag{2.15}$$

where $det(A)$ is the determinant of $\mathbf{A}$, given by

$$det(A) = \Gamma\kappa + g^2 - \Delta_a\Delta_c - i\left(\Delta_c\Gamma + \Delta_a\kappa\right). \tag{2.16}$$

Similarly one can derive equations for the expectation values of operator products. One uses

$$\frac{d}{dt}\langle\sigma^+ a\rangle = \langle\dot{\sigma^+} a\rangle + \langle\sigma^+ \dot{a}\rangle \tag{2.17}$$

so that one can substitute (2.10) into the equations above. Note that the ordering is important for the averages of the products involving noise operators. We introduce

$$\begin{aligned} X_1 &= a^+\sigma^- + \sigma^+ a \\ X_2 &= \frac{1}{i}\left(a^+\sigma^- - \sigma^+ a\right) \\ X_3 &= a^+ a \\ X_4 &= \sigma^+\sigma^- \end{aligned} \tag{2.18}$$

and derive the following equations for those operator products:

$$\langle\dot{\vec{X}}\rangle = \mathbf{B}\langle\vec{X}\rangle + \eta\langle\vec{I}\rangle, \tag{2.19}$$

where

$$\vec{X} := \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} -\gamma & -\Delta & 0 & 0 \\ \Delta & -\gamma & -2g & 2g \\ 0 & g & -2\kappa & 0 \\ 0 & -g & 0 & -2\Gamma \end{pmatrix}$$

$$\vec{I} = \begin{pmatrix} \sigma^- + \sigma^+ \\ \frac{1}{i}\left(\sigma^- - \sigma^+\right) \\ , a + a^+ \\ 0 \end{pmatrix} \tag{2.20}$$

with $\Delta = \Delta_a - \Delta_c$ and $\gamma = \Gamma + \kappa$.

Solving those equations for $\langle\dot{\vec{X}}\rangle = 0$ one obtains

$$\langle\vec{X}\rangle_0 = \frac{\eta^2}{|det(A)|^2}\left(2\Delta_a g, -2g\Gamma, \Delta_a^2 + \Gamma^2, g^2\right) \tag{2.21}$$

and by comparison with (2.15) one finds the equalities:

$$\begin{aligned}\langle a^+\sigma^-\rangle &= \langle a^+\rangle\langle\sigma^-\rangle \\ \langle a^+\ a\rangle &= \langle a^+\rangle\langle\ a\rangle.\end{aligned}$$

This is somewhat surprising as it shows that operator products factorize when the cavity is weakly driven. In general this is not the case and the entanglement between mode and atom produces e.g. additional terms in the diffusion. Let us take a look at the intracavity intensity. One finds from (2.21)

$$\langle a^+a\rangle = \eta^2\frac{\Delta_a^2+\Gamma^2}{\left(\Gamma\kappa+g(x)^2-\Delta_a\Delta_c\right)^2+\left(\Delta_a\kappa+\Delta_c\Gamma\right)^2}. \tag{2.22}$$

This shows that the intracavity intensity depends on the atomic position x. The atom introduces a shift in the cavity resonance frequency and therefore of the transmitted intensity [9]. From input-output formalism it is easy to find the following relations for the transmitted and reflected light for a cavity formed by two mirrors of equal loss rates κ [29, 30]:

$$\begin{aligned}\langle a\rangle_{ref} &= \frac{\eta}{\sqrt{2\kappa}}\frac{i\Delta_c}{2\kappa+i\Delta_c} \\ \langle a\rangle_{tra} &= \frac{\eta}{\sqrt{2\kappa}}\frac{-2\kappa}{2\kappa+i\Delta_c}.\end{aligned}$$

Thus for zero detuning ($\Delta_c = 0$) the transmitted intensity is maximal, whereas for large detunings most of the light is reflected back at the input mirror. When an atom is present in the cavity this simple rule is not valid any longer [9, 31].

Driving the cavity resonantly yields maximum intensity for the atom being at the nodes, as the cavity does not 'see' the atom there. Hence it does not experience any change in refractive index which would only shift it out of resonance and decrease the intracavity intensity (fig.2.2).

Quite opposite, if the cavity is driven out of resonance by an amount larger than the Rabi splitting $\Delta_c = -\Delta/2 - \sqrt{\Delta^2+4g_0^2}/2$ (see 4.2), the strong atom-mode interaction at the antinodes shifts the cavity back to resonance and therefore the intensity is largest at the antinodes (fig 2.3). This proves the strong atomic influence on the intracavity intensity.

From (2.21) it is also easy to calculate the force on the atom. As the part of the total Hamiltonian describing the mode-vacuum field interactions (V_{CV}) is space independent, its gradient equals zero. Analogous to the arguments of the preceding chapter one concludes that the gradient due to the radiated source field equals zero and the gradient of the free evolving field averaged over the

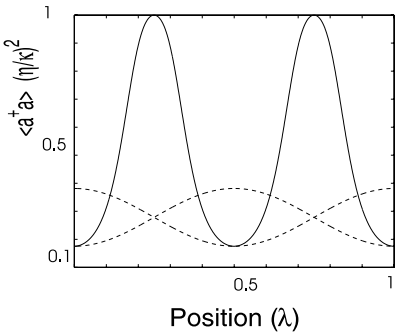

Figure 2.2: $\Delta_c = 0$ and so the cavity intensity (in units of $(\eta/\kappa)^2$) is maximal at the nodes. Note that the standing wave is represented by the dotted line for comparison. The averaged mean photon number $\overline{n} = 0.51(\eta/\kappa)^2$.

vacuum field yields zero. One is left with the contribution due to the atom-mode interaction V_{AC} (2.3) and finds for the dipole force operator in analogy to (1.15)

$$F(x) \quad = \quad -\hbar \nabla g \left(a^+ \sigma^- + \sigma^+ a \right) \tag{2.23}$$

and inserting the above results

$$\begin{aligned} f(x) &= Tr\left\{\rho F(x)\right\} \\ &= -\hbar\eta^2 \frac{\Delta_a \nabla g^2}{\left(\Gamma\kappa + g^2 - \Delta_a\Delta_c\right)^2 + \left(\Delta_a\kappa + \Delta_c\Gamma\right)^2}. \end{aligned} \tag{2.24}$$

Upon integration the corresponding potential reads:

$$\begin{aligned} V(x) &= \tfrac{-\eta^2\hbar\Delta_a}{d} Atan\left(\tfrac{u}{d}\right) \quad d \neq 0 \\ V(x) &= \tfrac{-2\eta^2\hbar\Delta_a}{u^3} \qquad\qquad\quad d = 0. \end{aligned}$$

Here u and d are the real and imaginary part of $det(A)$ respectively . One finds that for zero laser-atom detuning Δ_a the force is also zero. For negative detuning $\Delta_a < 0$ the atom is attracted to the antinodes 'high field seeker', whereas the opposite holds for positive detuning. Figures 2.4 show the steady state force and its potential for fixed $\Delta > 0$ where the cavity is once driven near resonance (low field seeker) and then the atom is driven slightly below resonance (high field seeker). We will come back to the steady state force and its potential later.

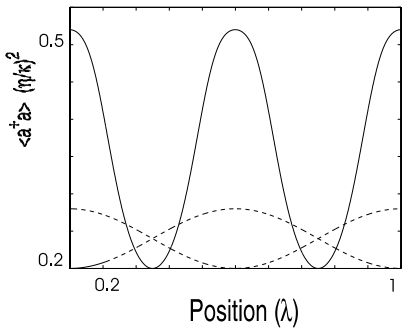

Figure 2.3: $\Delta_c > 0$ and the cavity is shifted into resonance at the antinodes.Note that the standing wave is represented by the dotted line for comparison. The averaged mean photon number $\overline{n} = 0.36(\eta/\kappa)^2$.

2.2 The friction force

It is straight forward and in complete analogy to chapter 1 eqn.(1.34) to find an expression for the friction coefficient, the linear velocity dependence of the force for small velocities. What is meant by 'small' velocities? There are two decay channels, one through the cavity and the other one through spontaneous emission. The velocity of the atom is certainly 'small', if it moves only a fraction of a wavelength before either decay may occur, i.e.

$$kv \ll \Gamma, \kappa. \tag{2.25}$$

It is generally not necessary for both ratios to be much smaller than one , depending on what decay channel is the predominant one and which type of cooling is chosen. For us it is the cavity decay which is of greater interest and the magintude of Γ is of little importance for the magnitude of velocities up to which a linear dependence is justified. We will come back to this in the next chapter.

For finding the friction coefficient one obtains the following equations:

$$v\frac{\partial}{\partial x}\langle\vec{Y}\rangle_0 = \mathbf{A}\langle\vec{Y}\rangle_1 \tag{2.26}$$

$$v\frac{\partial}{\partial x}\langle\vec{X}\rangle_0 = \mathbf{B}\langle\vec{X}\rangle_1 + \eta\langle\vec{I}\rangle_1, \tag{2.27}$$

where $\langle\vec{X}\rangle_0, \langle\vec{X}\rangle_1$ denote the zeroth order expectation values (calculated above) and first order expectation values (to be found!) respectively. The same applies to $\langle\vec{Y}\rangle_0$ and $\langle\vec{Y}\rangle_1$, as well as $\langle\vec{I}\rangle_1$. It does not present any difficulty

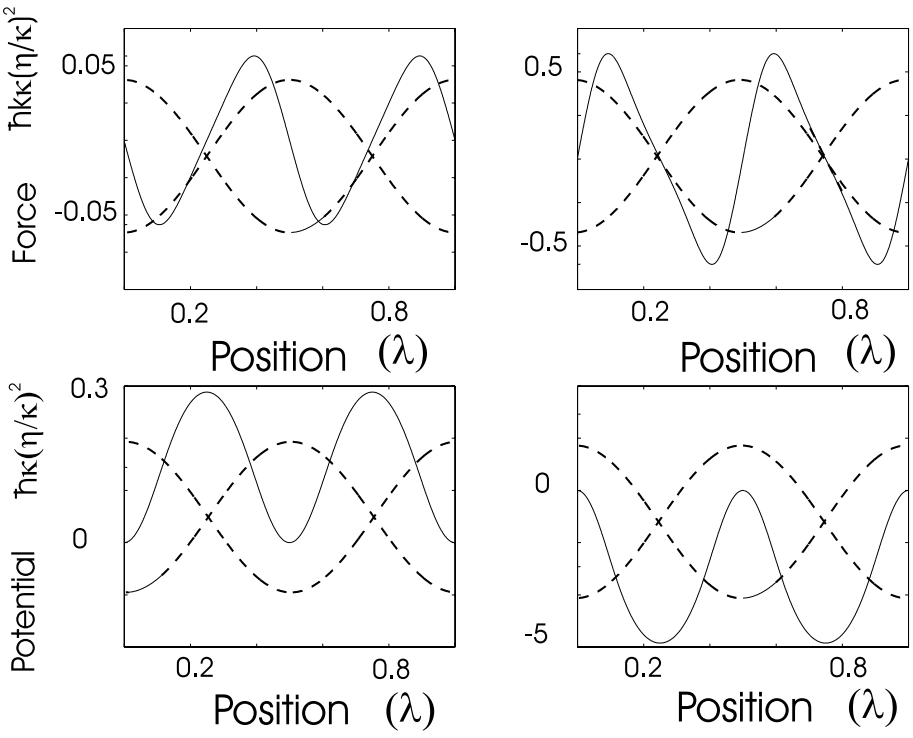

Figure 2.4: Dipole force acting on a motionless atom for $\Gamma = 2\kappa, g_0 = 4\kappa$ and fixed cavity-atom detuning $\Delta = 8\kappa$. In the left pictures $\Delta_a = -2\kappa$ (high field seeker) and in the right column $\Delta_a = 10\kappa$ (low field seeker). For comparison we have also plotted the field (dotted line).

either finding the derivative of the zeroth order expectation values nor inverting the two matrices $\mathbf{B}, \mathbf{A}$ again. [1] The friction force is simply written as

$$f_1(x) = -\hbar \nabla g \langle X_1 \rangle_1. \tag{2.28}$$

The expression found is somewhat long and complex and hence it is listed in the appendix A.1. We will not discuss it now but come back to it in the next sections.

2.3 Factorizing operator expectation values

Above we proceeded in an analytical way (in the limit of weak driving field) to obtain an expression for the friction force. However, for force exerted on an atom at rest we found that the $\langle \sigma^+ a \rangle$ operator product expectation values could be decorrelated at steady state. Can this also be done for a slowly moving atom? In order to check this one makes the following ansatz for the friction force:

$$f(x) = -\hbar \nabla g \left(\langle \sigma^+ \rangle \langle a \rangle + c.c. \right). \tag{2.29}$$

Now $\langle \sigma^+ \rangle, \langle a \rangle$ can be expanded in powers of v for small velocities like follows (in the above notation):

$$\begin{aligned} \langle \sigma^+ \rangle &= \langle \sigma^+ \rangle_0 + \langle \sigma^+ \rangle_1 \\ \langle a \rangle &= \langle a \rangle_0 + \langle a \rangle_1. \end{aligned}$$

Those expansions inserted into eqn.(2.29) yield a term zeroth order in v which is the steady state force of the motionless atom, a term second order in v which we have to omit in this approximation and a linear term as follows:

$$\begin{aligned} f_1(x) &= -\hbar \nabla g \left(\langle a \rangle_0 \langle \sigma \rangle_1 + c.c. \right) \\ &+ -\hbar \nabla g \left(\langle \sigma \rangle_0 \langle a_1 \rangle + c.c. \right). \end{aligned} \tag{2.30}$$

If one inserts the expressions for $\langle a \rangle_1, \langle \sigma^- \rangle_1$ found from eqns.(2.26), the expression of the friction force can be represented by two contributions

$$\begin{aligned} f_1(x) &= f_{ca} + f_{at} \\ &= -\hbar \nabla g \nabla \langle \sigma^- \rangle_0 T_{ca} + c.c. \\ &\quad -\hbar \nabla g \nabla \langle\ a \rangle_0 T_{at} + c.c., \end{aligned} \tag{2.31}$$

[1] Note that it is advantageous having a real matrix with the force operator expectation value being the first basis element of the matrix representation. This considerably simplifies inversion and later on calculation of force correlations.

where T_{ca}, T_{at} are not further specified linear combinations of $\langle a \rangle_0, \langle \sigma^- \rangle_0$. We will see that the two contributions to the force are the same one obtains through adiabatic elimination of mode (f_{at}) and atom (f_{ca}) respectively, which is discussed in the next chapter. The sum of both is found to be **equal** to the friction force found before in eqn(2.28). It thus seems that in the low velocity limit the **steady state** atom-mode operator products can be decorrelated (see also appendix C). Furthermore, the total friction force was separated into two contributions proportional to $\nabla \langle \sigma^- \rangle_0$ and $\nabla \langle a \rangle_0$ respectively. Those two contributions will be identified in the next chapter.

Chapter 3

Bad and good cavity limit

This chapter is dedicated to the adiabatic elimination of the field mode in the bad cavity limit or the atom in the good cavity limit respectively. It is helpful in distinguishing between a well known cooling mechansim (Doppler cooling) that is dominant in the bad cavity limit and the new one ('cavity cooling') that occurs because the cavity mode has its own dynamics. A simple explanation why Doppler cooling is prevalent in the bad cavity is that the situation resembles a free atom interacting with a standing wave (no cavity), where only Doppler cooling occurs.

3.1 Elimination of the cavity mode

We will now transform the master equation for atomic and mode operators into a master equation for atomic operators alone. This can be done by assuming that the cavity relaxation is much faster than the atomic relaxation so that the cavity adiabatically follows the atomic evolution. As a consequence one can get rid of the cavity operators and find a master equation for the atomic degrees of freedom alone. To derive it, it is not necessary to introduce a weak driving field approximation, as one starts from the Heisenberg equation for the mode operator, which is already linear. One makes the ansatz

$$\dot{a} = 0 = i\Delta_c a - ig(x)\sigma^- - \kappa a + \eta. \tag{3.1}$$

The noise operator F_1 can be omitted as it doesn't play any role when the mode is coupled to a vacuum field (T=0) and only normally ordered operator products are formed. One finds

$$a = \frac{\eta - ig(x)\sigma^-}{\kappa - i\Delta_c} \tag{3.2}$$

and similarly one can express a^+ and a^+a by atomic operators. These expressions can be resubstituted into eqns.(2.9) for $H_{J.C.}, H_P$ and after some algebra and omitting constant energy offsets in the Hamiltonian one obtains

$$\begin{aligned} H_{tot} &= H_{J.C.} + H_P \\ &= \hbar\Delta_c \frac{g^2}{\sqrt{\Delta_c^2+\kappa^2}}\sigma^+\sigma^- - \hbar\Delta_a\sigma^+\sigma^- + \hbar g(x)\left(\frac{\eta}{\kappa - i\Delta_c}\sigma^+ + h.c.\right) \\ &\quad -\hbar g(x)\kappa\frac{\eta}{\Delta_c^2+\kappa^2}\left(\sigma^- + \sigma^+\right). \end{aligned} \tag{3.3}$$

We still have to substitute the mode operators appearing in the Liouvillian describing the mode losses through cavity decay for eqn.(3.2)

$$\mathcal{L}_c = -\kappa\left(a^+a\rho + \rho a^+a - 2a\rho a^+\right) \tag{3.4}$$

which yields

$$\mathcal{L}_c = -\frac{\kappa g^2}{\Delta_c^2+\kappa^2}\left(\sigma^+\sigma^-\rho + \rho\sigma^+\sigma^- - 2\sigma^-\rho\sigma^+\right). \tag{3.5}$$

This describes the cavity decay expressed through atomic operators
We also find an **additional** contribution to the atomic Hamiltonian

$$H_{add.} = +\hbar g(x)\kappa\frac{\eta}{\Delta_c^2+\kappa^2}\left(\sigma^- + \sigma^+\right) \tag{3.6}$$

It cancels the last term of the Hamiltonian (3.3) above. Adding all contributions from atomic decay, cavity decay and the different Hamiltonians one finds the following master equation for the atomic operators:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}\left[H_{at}, \rho\right] + \mathcal{L}\rho, \tag{3.7}$$

where

$$H_{at} = -\hbar\sigma^+\sigma^-\left(\Delta_a - \Delta_c\frac{g(x)^2}{\Delta_c^2+\kappa^2}\right) + \hbar g\frac{\eta}{\sqrt{\Delta_c^2+\kappa^2}}\left(\sigma^+ + \sigma^-\right) \tag{3.8}$$

$$\mathcal{L}\rho = -\left(\Gamma + \kappa\frac{g(x)^2}{\Delta_c^2+\kappa^2}\right)\left(\sigma^+\sigma^-\rho + \rho\sigma^+\sigma^- - 2\sigma^-\rho\sigma^+\right). \tag{3.9}$$

It would be interesting to examine this master equation further, e.g. modified spontaneous emission and things related to that. Here we will primarily concentrate on the induced light forces.

The general expression for the force with the mode adiabatically eliminated reads:

$$
\begin{aligned}
F(x) &= -\frac{\partial H_{at}}{\partial x} \\
&= -\hbar\Delta_c \frac{\nabla g(x)^2}{\Delta_c^2+\kappa^2}\sigma^+\sigma^- - \hbar\nabla g(x)\frac{\eta}{\sqrt{\Delta_c^2+\kappa^2}}\left(\sigma^+ + \sigma^-\right). \quad (3.10)
\end{aligned}
$$

The master equation (3.7) is valid for any pumping rate η and eqn.(3.10) yields the corresponding force. To find the exact force without the 'weakly driven cavity' approximation we note that the master equation (3.7) can be defined by

$$
\begin{aligned}
H_{at} &= -\hbar\sigma^+\sigma - \tilde{\Delta} + \hbar\tilde{g}(x)\left(\sigma^+ + \sigma -\right) \\
\mathcal{L} &= -\tilde{\Gamma}\left(\sigma^+\sigma^-\rho + \rho\sigma^+\sigma^- - 2\sigma^-\rho\sigma^+\right), \qquad (3.11)
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{\Delta} &= \Delta_a - \Delta_c\frac{g(x)^2}{\Delta_c^2+\kappa^2} \\
\tilde{g} &= \frac{g(x)\eta}{\sqrt{\Delta_c^2+\kappa^2}} \\
\tilde{\Gamma} &= \Gamma + \kappa\frac{g(x)^2}{\Delta_c^2+\kappa^2}.
\end{aligned}
$$

Noting the analogy to (1.21) we can immediately take the results (1.24) for u_x and ρ_{ee} of the optical Bloch equations and insert them to find for the force

$$
\begin{aligned}
f(x) &= Tr\left\{\rho F(x)\right\} \\
&= -\hbar\Delta_c\frac{\nabla g(x)^2}{\Delta_c^2+\kappa^2}\frac{1}{2}\frac{s}{s+1} \\
&\quad -\hbar\nabla g(x)\frac{\eta}{\sqrt{\Delta_c^2+\kappa^2}}\frac{\tilde{\Delta}}{2\tilde{g}(x)}\frac{s}{s+1}, \qquad (3.12)
\end{aligned}
$$

where s is defined as

$$
s = \frac{2\tilde{g}(x)^2}{\tilde{\Delta}^2+\tilde{\Gamma}^2}.
$$

Note again, that expression (3.12) is valid for any driving amplitude η. It can be shown by expanding (3.12) in powers of η^2 that the expression for the force we found before in (2.24) is just the contribution proportional to η^2 of (3.12).

Let us check, if we obtain the same result by linearisation of the master equation. As we want to discuss the weakly driven cavity it is possible to linearise (3.7) for weak pumping and to obtain the corresponding Heisenberg equation (with the noise terms omitted) for the atomic operator σ^- :

$$\dot{\sigma^-} = i\left(\Delta_a - \Delta_c \frac{g(x)^2}{\Delta_c^2+\kappa^2}\right)\sigma^- - \left(\Gamma + \kappa\frac{g(x)^2)}{\Delta_c^2+\kappa^2}\right)\sigma^- - ig(x)\frac{\eta}{\sqrt{\Delta_c^2+\kappa^2}}. \tag{3.13}$$

As we showed in the preceding chapter, for weak pumping (linearised Heisenberg equations as as consequence) the $\langle\sigma^+\sigma^-\rangle$ steady state expectation values factorize:

$$\langle\sigma^+\sigma^-\rangle = \langle\sigma^+\rangle\langle\sigma^-\rangle. \tag{3.14}$$

The same of course applies here and this enables us to find the force by calculating the σ^- expectation values for $v = 0$ or very small velocities. Taking the expectation values over eqn.(3.13) and setting the rigth side equal to zero one obtains $\langle\sigma^-\rangle_0$, and using relation (3.14) for the factorization as well as eqn.(3.10) for the force one obtains the result for the force on the motionless atom found before in (2.24).

Let us now can consider the friction coefficient proceeding anlogously to section 2.3. One finds the contribution f_{at} to the total friction, introduced before in (2.31) and listed in the appendix (A.2). The necessary equations describing the calculation of the first order term which has to be substituted into the expression for the force are:

$$\begin{aligned} v\nabla\langle\sigma^-\rangle_0 &= \left(i\tilde{\Delta} - \tilde{\Gamma}\right)\langle\sigma^-\rangle_1 \\ f_{at} &= -\hbar\Delta_c\frac{\nabla g(x)^2}{\Delta_c^2+\kappa^2}\left(\langle\sigma^+\rangle_1\langle\sigma^-\rangle_0 + \langle\sigma^+\rangle_0\langle\sigma^-\rangle_1 + c.c.\right) \\ &\quad -\hbar\nabla g(x)\frac{\eta}{\sqrt{\Delta_c^2+\kappa^2}}\left(\langle\sigma^+\rangle_1 + c.c.\right). \end{aligned}$$

3.1.1 Velocity independent intensity

For standard Doppler cooling in a standing wave, the electric field is assumed unaffected by the presence of the atom. In our model the intensity of the standing wave is strongly dependent on the atomic position. Mechanical light effects of a two level system interacting with a cavity mode have been investigated before in a paper by Doherty et al. [32]. For comparison we want to take a close look at this paper now. To simplify calculating efforts they suggested the following approximations to the problem at hand:

i.) The evolution of mode operators is very fast and the mode does not only follow the atom adiabtically but to a first approximation the mode can also be assumed to be independent of the atomic velocity (This is the ultimate limit of our adiabatic approximation when the mode is completely decoupled from the dynamics.)

ii.) The field inside the cavity is more or less coherent and thus atom-mode operator products factorize (coherent approximation).

So they replaced the mode operators by c-numbers which were calculated for the steady state and $v = 0$ from the master equation. The cavity loss and the driving term appearing in the master equation can then be omitted and the coupling parameter in the Jaynes Cummings interaction term is replaced by

$$\omega_0 \to \omega_0 |\langle a \rangle_0|. \tag{3.15}$$

One is left with the optical Bloch equations (1.23) for which analytical expressions can be found for f(v=0) (1.32), for the friction force (1.36) and for the diffusion (1.45). Let us check how those results compare to what we find in the exact description All we do is take the previously calculated results for ($s \ll 1$) of the Bloch equations and use (3.15).

For the force at low saturation we find from (1.32)

$$\begin{aligned} f(x) &= -\frac{\hbar\Delta_a}{2}\frac{\nabla\omega_R^2/2}{\Delta_a^2+\Gamma^2} \\ &= -\frac{\hbar\Delta_a}{2}\frac{\nabla cos^2(kx)\omega_0^2/2}{\Delta_a^2+\Gamma^2}\frac{\Delta_a^2+\Gamma^2}{|det(A)|^2} \\ &= -\hbar\eta^2\frac{\Delta_a \nabla g^2}{|det(A)|^2}, \end{aligned} \tag{3.16}$$

which is just the expression for the steady state force found before in (2.24). So for the atom at rest the results are the same what was to be expected as the dynamics does not enter. For the diffusion we find analogously starting from (1.45) and substituting (3.15):

$$\begin{aligned} D_{tot} &= \hbar^2 k^2 g^2 \frac{\eta^2\Gamma}{|det(A)|^2} \\ &\quad +\hbar^2 (\nabla g)^2 \frac{\eta^2\Gamma}{|det(A)|^2} \\ &= \hbar^2 k^2 \frac{g_0^2\eta^2\Gamma}{|det(A)|^2}. \end{aligned} \tag{3.17}$$

This result does not hold in general as we will see later (5.23). It does not consider the diffusion caused by atom-mode interactions and omits an important

term to the total result. Now what about the friction force? We can take expression (1.36), do the replacement (3.15) again.We get

$$f_1(x) = v\hbar k^2 g_0^2 4 sin^2 kx \frac{\Delta_a \Gamma}{\Delta_a^2 + \Gamma^2} |det(A)|^2 \tag{3.18}$$

and averaging the resulting force over a wavelength to obtain the friction coefficient we only find agreement to the friction force obtained from (2.28) when $\kappa \gg \Gamma, g$, that is when the atom is 'quasi free'. In this limit the results actually match perfectly and fig 3.1 shows the typical Doppler dispersion profile of the averaged friction force along the Δ_a axis, whereas hardly any variation occurs along the Δ axis. Note that cooling occurs for $\Delta_a < 0$.

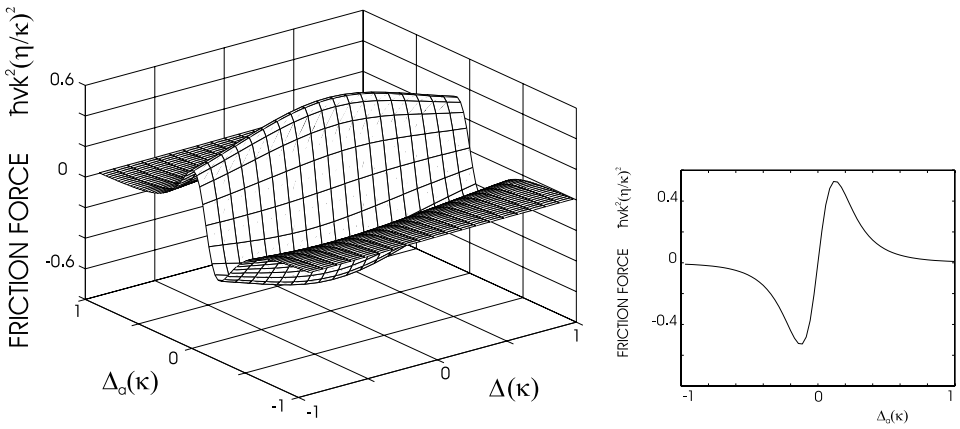

Figure 3.1: For $\kappa \gg \Gamma, g$ the friction force is given by f_{at} which is related to Doppler cooling. The left figure displays a plot of f_{at} against Δ, Δ_a and the right one gives a crosscut for $\Delta = 0$ where f_{at} is maximal.

In summary, the force obtained through the recipe (3.15) suggested by Doherty et al. turns out to be only valid in the limit $\kappa \gg \Gamma, g$. In this limit f_{at} is also a correct expression for the total friction force (2.28).

We may ask ourselves when the adiabatic approximation holds after all. Comparison between the total friction force f_1 of eqn.(2.28) and the Doppler cooling force f_{at} for different sets of parameters shows that

$$\left.\begin{array}{r} \sqrt{\Delta_c^2 + \kappa^2} \gg g \\ \Delta_a \approx 0 \end{array}\right\} \rightarrow f_1 \approx f_{at}, \tag{3.19}$$

that is when the atom is driven more or less resonantly and when the cavity is far detuned and decays much quicker than a period of the Rabi oscillations between ground and excited state. Note that f_{at} is even the dominant contribution when κ is small if the detuning Δ_c is big enough and (3.19) becomes exact when $\Delta_a = 0 : \ f_1 = f_{at}$ for $\Delta_a = 0$.

3.2 Elimination of the atomic operators

In the opposite limit of a very good cavity one can assume the internal atomic dynamics to be much faster than the cavity dynamics. Proceeding analogously to the preceding section in this limit one makes the ansatz

$$\dot{\sigma^-} = 0$$

and obtains[1] from the second (linearised) Heisenberg equation (2.10)

$$\sigma^- = \frac{-ig(x)a}{\Gamma - i\Delta_a}. \tag{3.20}$$

From this one can express $\sigma^-, \sigma^+\sigma^-$ through mode operators and substitute it into the master equation (2.8). The total Hamiltonian turns out to be (omitting constant energy shifts):

$$\begin{aligned} H_{ca} &= H_{J.C.} + H_P \\ &= -\hbar a^+ a \left(\Delta_c - \Delta_a \frac{g(x)^2}{\Delta_a^2 + \Gamma^2}\right) - i\eta \left(a - a^+\right). \end{aligned} \tag{3.21}$$

The term in the master equation (2.8) describing atomic spontaneous decay

$$\mathcal{L}_{SE}\rho = -\Gamma \left(\sigma^+\sigma^-\rho + \rho\sigma^+\sigma^- - 2\sigma^-\rho\sigma^+\right)$$

must also be expressed by a, a^+ which yields upon substitution of (3.20)

$$\mathcal{L}_{SE}\rho = -\Gamma \frac{g(x)^2}{\Delta_a^2 + \Gamma^2} \left(a^+ a\rho + \rho a^+ a - 2a\rho a^+\right)$$

and so for the complete master equation we have

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}\left[H_{ca}, \rho\right] + \mathcal{L}\rho \tag{3.22}$$

with H_{ca} given above (3.21) and

$$\mathcal{L}\rho = -\left(\kappa + \Gamma \frac{g(x)^2}{\Delta_a^2 + \Gamma^2}\right) \left(a^+ a\rho + \rho a^+ a - 2a\rho a^+\right). \tag{3.23}$$

Equivalently the Heisenberg equation (with noise term omitted) can be written as:

$$\dot{a} = \left(-\kappa - \gamma\left(x\right) + i\Delta_c - iU\left(x\right)\right) a + \eta. \tag{3.24}$$

[1] Here again the noise operators are omitted as they cancel when averages are calculated and operator products are written in normal ordering.

with

$$\gamma(x) = \Gamma \frac{g\,(x)^2}{\Delta_a^2 + \Gamma^2}$$
$$U(x) = \Delta_a \frac{g\,(x)^2}{\Delta_a^2 + \Gamma^2}.$$

For the force one finds

$$\begin{aligned} F(x) &= -\frac{\partial}{\partial x} H_{ca} \\ &= -a^+ a \frac{d}{dx} U(x). \end{aligned} \tag{3.25}$$

Now one can calculate the steady state force for $v = 0$ or v very small assuming

$$\langle a^+ a \rangle = \langle a^+ \rangle \langle a \rangle$$

like before. Again one finds for the force on the motionless atom eqn.(2.24). For very small velocities one proceeds analogously to section 2.3 and finally obtains the component f_{ca} introduced in (2.31) and listed in the appendix (A.3). The necessary equations are:

$$\begin{aligned} v\nabla\langle a\rangle_0 &= (-\kappa - \gamma(x) + i\Delta_c - iU(x))\,\langle a\rangle_1 \\ f &= -\left(\langle a^+\rangle_0 \langle a\rangle_1 + \langle a^+\rangle_1 \langle a\rangle_0 + c.c.\right) \frac{d}{dt} U(x). \end{aligned}$$

So together with the last section one has derived the relation for the friction force

$$f_1(x) = f_{at} + f_{ca} \tag{3.26}$$

of section 2.3 with f_{at}, f_{ca} indentified as the Doppler cooling ($\dot{a} = 0$) and 'cavity cooling' ($\dot{\sigma^-} = 0$) forces respectively. One can analogously to (3.19) find the parameters where the 'cavity cooling' force is the dominant one. Those are found to be

$$\left.\begin{array}{r} \sqrt{\Gamma^2 + \Delta_a^2}, g \gg \kappa \\ \Delta_c \approx 0 \end{array}\right\} \rightarrow f_1 \approx f_{ca}. \tag{3.27}$$

This is the case when the cavity is more or less in resonance with the pump and when spontaneous decay or detuning from the atomic transition are much

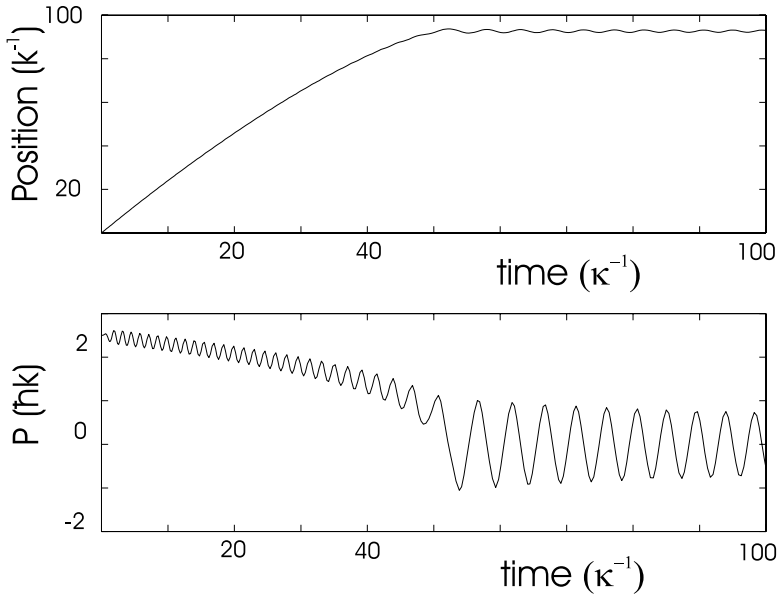

Figure 3.2: The particle is cooled while moving along the laser standing wave until its kinetic enrgy gets so low that it is trapped in a potential well where oscillates back and forth.

larger than cavity decay. It is also essential for the Rabi oscillations to be larger than cavity decay as otherwise conditon (3.19) implies strong influence of the Doppler cooling force f_{at}. For **high velocities** the force contributions obtained from adiabtically eliminating the atom or the mode don't necessarily sum up to obtain the total force as 'cross terms' may occur, but for **small velocities** we have shown they do. Also the operator products can't necessarily be decorrelated any more and only in the case of the cavity being in a coherent state the dynamical equations for the mode and atomic motion (3.24),(3.25) can be written as

$$\begin{aligned} \dot{\alpha} &= (-\kappa - \gamma(x) + i\Delta_c - iU(x))\,\alpha + \eta \\ \dot{p} &= -|\alpha|^2 \frac{d}{dx} U(x) \\ \dot{x} &= \frac{p}{m}. \end{aligned} \tag{3.28}$$

This is a purely classical set of equations with the atomic internal degrees of freedom adiabitcally eliminated, the mode replaced by a c-number and the

pumping assumed to be weak. Still, with the choice of certain parameters , where f_{ca} is by far bigger than f_{at} , the model gives the exact cooling force for small velocities and for large velocities it is a good approximation as well. This equation can be easily generalised to three dimensions. Integration of (3.28) gives a useful physical picture of the particle being cooled down until it is trapped in a periodic potential. This is shown in figure 3.2, where a particle moves along a laser standing wave with detunings and decay rates chosen such that the main contribution comes from the 'cavity cooling' force. Note that it does not take into account diffusion of any kind nor does it include Doppler cooling, which is, however, small for the chosen parameters. In the following we will give two useful interpretations for this force.

3.2.1 Dressed state interpretation

Note that much of what follows is in direct connection to the next chapter on 'dressed states' where many of the new concepts are treated in more detail. Similarly to the picture given by Cohen Tannoudji for the movement of an atom in a standing wave [11], one can consider the particle moving in the potential $U(x)$ with a space dependent decay rate given by $\kappa + \gamma(x)$. This potential corresponds to the dressed state which consists to the biggest part of $|g,1\rangle$ (see section 4.4). So the primary decay is over the cavity κ which is also pumped close to resonance. As the atom is interacting with the mode which is described by the coupling g, the atom is moving in a combined state containing a small admixture of the excited state $|e\rangle$, which can decay spontaneously depending on the particle position. The cavity decay does not depend on the position as the combined state remains to the most part the ground state at all positions. Assuming $\Delta_c = 0$, that is the cavity is driven resonantly, and $\Delta > 0$, that is $\omega_c > \omega_{10}$, the atom is preferentially pumped into the nodes of the combined state which varies according to $U(x)$. When the particle moves slowly $\left(\frac{kv}{\kappa} \ll 1\right)$ that it decays through the cavity before it reaches the 'top of the hill', it loses some of its kinetic energy. This scheme is called Sisyphus cooling and has been treated in detail in connection with atomic movement in a strong standing wave [11] or polarization gradient cooling [33] before. The sign of Δ_a and the constant energy offset Δ_C are determining whether it is 'up' or 'down' that the atom is moving and thus whether it is gaining or losing kinetic energy. The analogies for $\Delta_a \gg g$ between the dressed state model and the adiabatic model presented in this chapter will be shown in section 4.4. Fig.3.4 gives a representation of the Sisyphus cooling scheme when the atom is pumped to the antinodes of the lower dressed level. This is also confirmed by fig. 3.5 and 3.6, where the averaged friction force is plotted against the detunings Δ_a, Δ_c for $g > \Gamma, \kappa$. As one can see, there are two main areas of cooling, marked on the contour plot. The area along

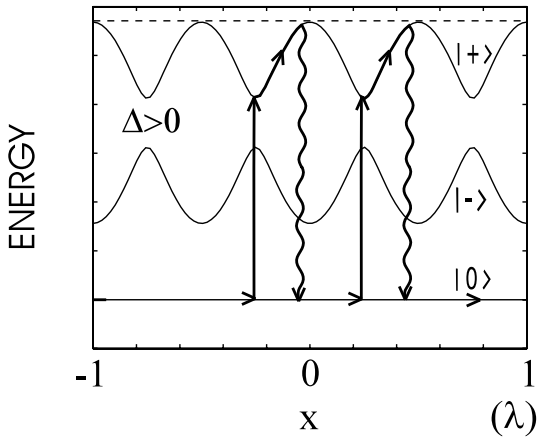

Figure 3.3: $\omega_c > \omega_{10}$: The atom is pumped to the node of the upper dressed level from where it moves upwards and decays back to the flat ground level.

$$\Delta_c = 0 \qquad \Delta > 0 \tag{3.29}$$

corresponds to driving the cavity resonantly , i.e. pumping the atom into the nodes of the upper dressed level, from where it loses kinetic energy on the way up. This is shown schematically in fig.3.3. The second area is is around

$$\Delta_c = -\Delta/2 - \sqrt{\Delta^2 + 4g_0^2}/2 \qquad \Delta < 0, \tag{3.30}$$

which corresponds to pumping the antinodes of the lower dressed level. This is shown in fig.3.4. Note that in both cases the level pumped is the one that turns into $|g,1\rangle$ for zero atom-mode coupling ($g = 0$) and contains only small admixtures of $|e,0\rangle$ at the antinodes. The reason why similar features do not appear when we pump the other level is given in section 4.3.

The 'heating' features can be explained analogously. The very good agreement to the predictions of the dressed state scheme is again found in fig.3.7 for similar parameters, where the thick lines are drawn according to

$$\Delta_c = -\Delta/2 - \sqrt{\Delta^2 + 4g_0^2}/2 \qquad \Delta < 0$$
$$\Delta_c = -\Delta/2 + \sqrt{\Delta^2 + 4g_0^2}/2 \qquad \Delta > 0,$$

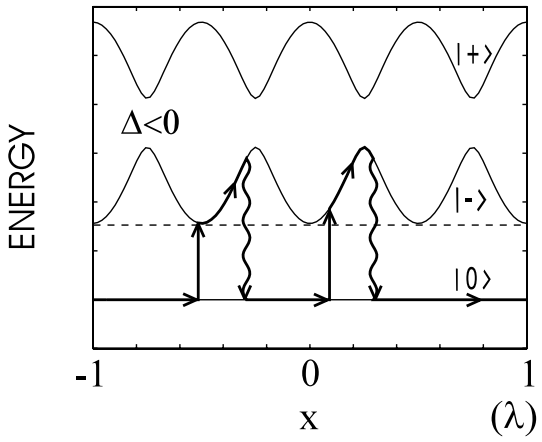

Figure 3.4: Sisyphus cooling scheme for $\omega_c < \omega_{10}$: The atom is pumped to the antinode of the lower dressed level from where it moves upwards and decays back to the flat ground level.

which correspond to pumping at the antinodes to obtain cooling for ($\Delta < 0$) or heating for ($\Delta > 0$). It shows that the contours of the friction force features match with the contours suggested by the dressed state model.

What about the two peaks around the origin? They result from Doppler cooling and can be obtained independently by using (3.18). This force appears at very small detunings whereas the dressed state force appears for large detunings. This is because the splitting between the levels must be large enough in order to prevent coherences and populating the other dressed level so that the simple picture of an atom moving in a potential is valid . On the other hand, increasing the splitting also means decreasing the height of the potential wells which goes like $\frac{g_0}{\Delta}$ for large Δ. Thus maximum cooling is achieved somewhere in between the two extremes where coherences are still moderately small and the potential wells high enough.

One can see in the spatial variation of the friction force in fig.3.8, that we are not entirely in the limit where adiabatic elimination of the atom holds (3.27). There is still a significant contribution due to the Doppler effect, which produces an antidamping force at the nodes, whereas the 'cavity cooling' force f_{ca} alone equals zero at the nodes. This can be suppressed by choosing larger g and Δ. Note that the additional Doppler contribution to the Sisyphus force is a cooling one when $\Delta < 0$ and the antinodes of the lower dressed level are pumped as in fig.3.4. This is because the laser is tuned below the atomic resonance and

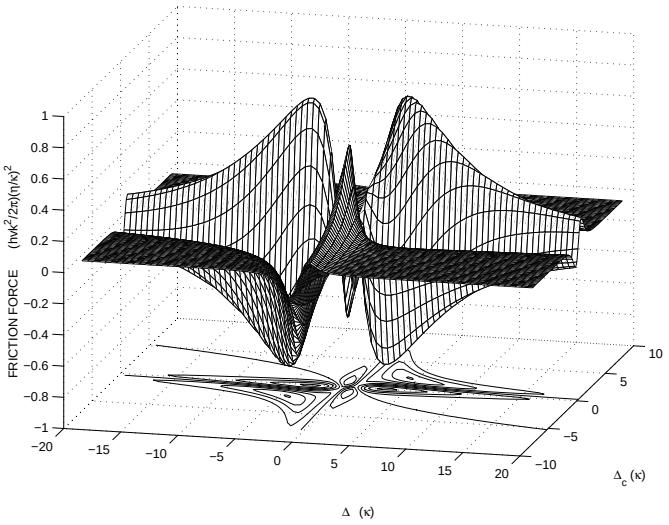

Figure 3.5: $\Gamma = \kappa, g_0 = 3\frac{1}{3}\kappa$. The averaged friction force is plotted against the detunings Δ_a, Δ_c. The longstretched features can be explained by the Sisyphus scheme and the two sharp peaks at the origin by Doppler cooling.

so *Doppler cooling* occurs. However, for $\Delta > 0$ and pumping of the nodes of the upper level as suggested in fig.3.3, the additional Doppler contribution is an antidamping one. This is because the laser is tuned above atomic resonance and so *Doppler heating* occurs, which slightly reduces the Sisyphus cooling effect as can be seen in fig.3.8.

In the range of parameters given by (3.27) The splitting between the dressed states can be much larger or smaller than the linewidth of the atomic transition Γ. In fig. 3.5 we considered the case $g > \Gamma$. If $\Gamma > g$ the dressed state character of the friction force disappears for $\Delta_a < \Gamma$ as the two dressed levels are not clearly separated. Coherences build up and destroy the Sisyphus cooling scheme. The 'sidefeatures' arising from driving the cavity at the antinodes become smaller and pushed back into a region where $\Delta_a > \Gamma$ and the dressed levels are separated again. Furthermore, the Doppler peaks of fig.3.5 have completely disappeared. This can be seen in fig.3.9

3.2.2 Dynamical intensity interpretation

Let us now look at the same model from a different perspective. In principle it is the same argument as above. The cavity intensity is strongly dependent

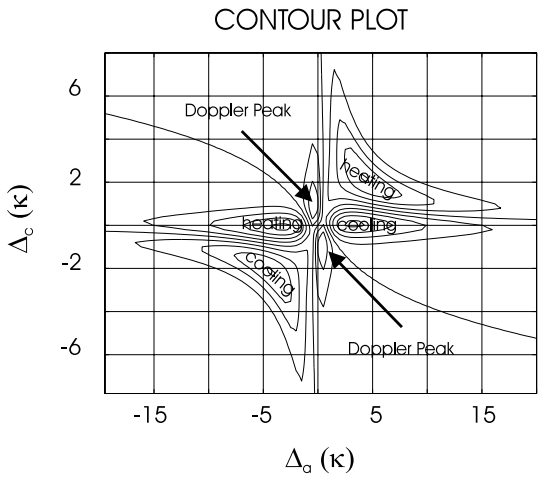

Figure 3.6: contourplot to fig. 3.5

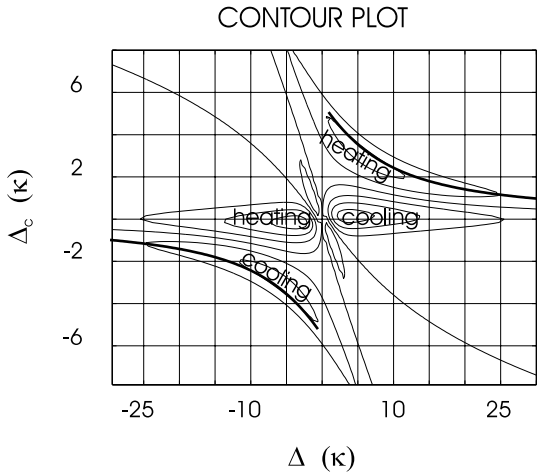

Figure 3.7: Contour plot of the friction force for $\Gamma = \kappa, g_0 = 5.5\kappa$. The bold lines indicate the pumping of the antinodes of the lower(left) and the upper(right) antinode.

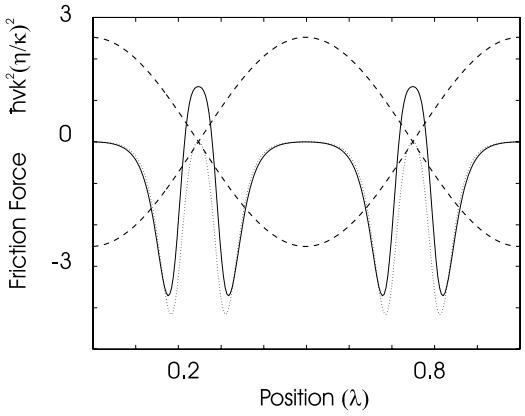

Figure 3.8: The spatial dependence of the total friction force and the 'cavity cooling' force f_{ca} (dotted line) for Γ, κ, g_0 like in fig. 3.5. $\Delta_c = 0$ and $\Delta = 3\kappa$ yielding the maximal friction force were chosen.

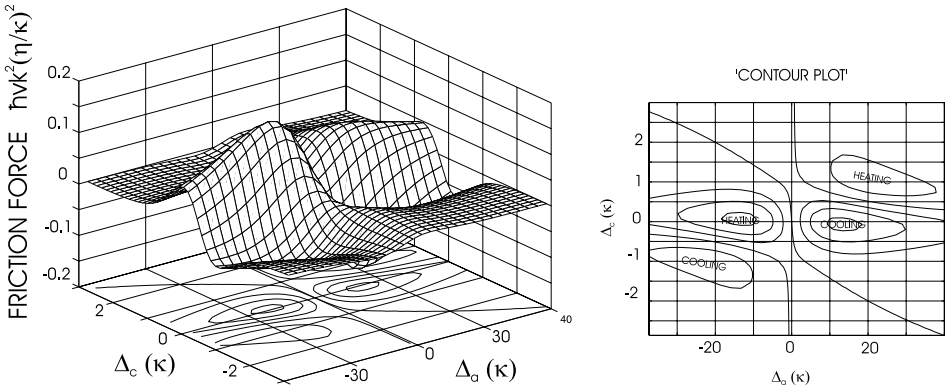

Figure 3.9: The friction force (mainly f_{ca}) for $\Gamma = 12\kappa, g_0 = 4\kappa$. Note that the contribution from pumping the nodes is very small and only gets larger again when Δ_a and hence splitting between the dressed levels increases.

on the atomic position (2.22). This is because the particle shifts the cavity resonance frequency considerably, which is only possible if the coupling between cavity mode and particle is strong enough $g \gg \kappa$ (see also 3.27). For certain parameters (i.e. $\Delta_a \gg g, \Delta_c = 0$), the intensity is a maximum at the nodes of the standing wave whereas the potential $U(x)$ is a minimum there. However, if the particle is moving slowly along the potential $U(x)$ the maximum field intensity will be reached after it has passed the potential minimum due to the finite cavity response time. Accordingly, the particle sees a higher intensity and thus a stronger damping force when going up $U(x)$ than it sees an antidamping force when going down $U(x)$. On average this leads to a damping force.

3.2.3 The cooling force for arbitrary velocities

It is possible to integrate (3.28) considering periodic boundary conditions.

$$v\frac{\partial \alpha}{\partial x} = D(x)\alpha + \eta, \tag{3.31}$$

where $D(x)$ is given by (3.28). Solving this linear differential equation for x yields

$$\alpha(x) = m(x_0, x)\alpha(x_0) + \int_{x_0}^{x} dx' m(x', x)\frac{\eta}{v}, \tag{3.32}$$

where

$$m(x_0, x) = e^{\int_{x_0}^{x} dx'' \frac{D(x'')}{v}}. \tag{3.33}$$

Making use of the periodic boundary conditions, we set

$$x_0 = x - \lambda$$

and obtain

$$\alpha(x) = \frac{\int_{x-\lambda}^{x} dx' m(x', x)\frac{\eta}{v}}{1 - m(x - \lambda, x)}. \tag{3.34}$$

The indefinite integral over $D(x)$ can be found analytically and so one obtains after some simplifications and algebra

$$\alpha(x) = \frac{\frac{\eta}{v} g(x) G(-x, \lambda - x)}{1 - g(x)g(\lambda - x)}, \tag{3.35}$$

where

$$g(x) = e^{\frac{x}{v}\left(-\kappa + i\Delta_c - \frac{g_0^2}{2}\frac{\Gamma + i\Delta_a}{\Delta_a^2 + \Gamma^2}\right) - \frac{sin(2x)}{v}\frac{g_0^2}{4}\frac{\Gamma + i\Delta_a}{\Delta_a^2 + \Gamma^2}} \tag{3.36}$$

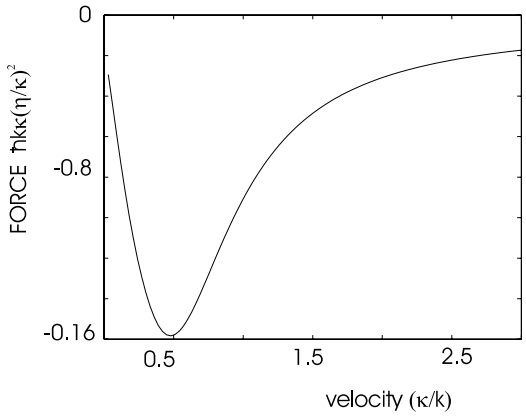

Figure 3.10: Velocity dependence of the adiabatic (internal dynamics of the atom eliminated) cooling force for $\Gamma = 2\kappa, g_0 = 4\kappa, \Delta_c = 0, \Delta_a = 12\kappa$.

and

$$G(x_1, x_2) = \int_{x_1}^{x_2} dx g(x).$$

Fig. 3.10 shows the force plotted against the velocity for typical parameters. One clearly sees the linear increase for $\frac{kv}{\kappa} \ll 1$. At these velocities the atom only covers a small distance before it decays over κ. For higher velocities the decay is averaged over many wavelenghts which yields a hyperbolic decrease of the force. This curve can be seen in analogy to the velocity dependence of the Sisyphus effect of an atom moving in a strong classical standing wave as derived by Cohen Tannoudji et al. [11].

Chapter 4

The dressed state model

In this chapter we want to discuss the nice interpretation of the new 'cavity cooling' force found in the previous chapter using dressed states. Although it is analytically tiresome expressing the master equation in the dressed basis [34, 11], the efforts are justified by the simple physical picture one finally obtains. A calculation of the friction coefficient using the dressed master equation, which does not provide new physical insights, but allows a comparison to the results derived in chapter 1, is put into the appendix.

4.1 Dressed states of the atom-field Hamiltonian

The total Hilbertspace of the system $\mathcal{H}_{tot}$ may be represented as the product space of the atom state space and the field state space, i.e. $\mathcal{H}_{tot} = \mathcal{H}_{atom} \otimes \mathcal{H}_{field}$. The atom may be represented in the $\{|e\rangle, |g\rangle\}$ basis, corresponding to the excited state and the ground state respectively. For the field the single mode Fock states $\{|n\rangle\}$ form a convenient basis, with 'n' denoting the number of photons in the single mode of frequency ω_c. Thus the tensor product space may be represented in the $\{|e,n\rangle, |g,n\rangle\}$ basis. $H_{J.C.}$ splits the state space into different manifolds $\mathcal{E}_n$ with each manifold spanned by the vectors $\{|e,n\rangle, |g,n+1\rangle\}$ for a fixed 'n'. The Hamiltonian may now be represented in that basis and diagonalized to obtain the eigenvectors of the manifold $\mathcal{E}_n$. From now on we set $\hbar = 1$ for convenience and introduce the resonant Rabi frequency $\omega_R = 2g\sqrt{n+1}$:

$$H_{J.C.} = \begin{pmatrix} -\Delta_a - n\Delta_c & \frac{\omega_R}{2} \\ \frac{\omega_R}{2} & -(n+1)\,\Delta_c \end{pmatrix} =$$

$$\begin{pmatrix} -\Delta_c\,(n+1) - \frac{\Delta}{2} & 0 \\ 0 & -\Delta_c\,(n+1) - \frac{\Delta}{2} \end{pmatrix} - \frac{\Delta}{2} \begin{pmatrix} 1 & \frac{-\omega_R}{\Delta} \\ \frac{-\omega_R}{\Delta} & -1 \end{pmatrix}. \tag{4.1}$$

On diagonalising the above matrix, one finds the eigenvalues corresponding to the upper and the lower eigenenergy respectively:

$$\begin{aligned} \lambda_+ &= -\,(n+1)\,\Delta_c - \frac{\Delta}{2} + \frac{\Omega_n}{2} \\ \lambda_- &= -\,(n+1)\,\Delta_c - \frac{\Delta}{2} - \frac{\Omega_n}{2}, \end{aligned} \tag{4.2}$$

where the effective Rabi frequency

$$\Omega_n = \sqrt{\Delta^2 + 4g_n^2} \tag{4.3}$$

and the atom-cavity detuning

$$\Delta = \Delta_a - \Delta_c = \omega_c - \omega_{10}$$

were introduced. The eigenvalues yield the following normalized eigenfunctions (the index n of Ω_n has been omitted):

$$\begin{aligned} |+,n\rangle &= \sigma\sqrt{\frac{\Omega-\Delta}{2\Omega}}|e,n\rangle + \sqrt{\frac{\Omega+\Delta}{2\Omega}}|g,n+1\rangle \\ |-,n\rangle &= \sqrt{\frac{\Omega+\Delta}{2\Omega}}|e,n\rangle + \sigma\sqrt{\frac{\Omega-\Delta}{2\Omega}}|g,n+1\rangle. \end{aligned} \tag{4.4}$$

The σ introduced above stands for $sign\,(\omega_R)$ and (as ω_R is a periodic function in the wavelength λ) jumps accordingly between -1 and $+1$. For $\Delta > 0$, $|+,n>$ and $|-,n>$ as introduced above in eqns. 4.4 can be differentiated with respect to space. For $\Delta < 0$, above equations must be multiplied by σ — preserving the normalization requirement — to obtain differentiability. This shall be important later. It turns out to be convenient to use trigonometric functions and express above equations in a simpler form. We introduce

$$\begin{aligned} sin\theta_n = \sqrt{\tfrac{\Omega+\Delta}{2\Omega}} \quad cos\theta_n = \sigma\sqrt{\tfrac{\Omega-\Delta}{2\Omega}} \qquad \Delta > 0 \\ sin\theta_n = \sigma\sqrt{\tfrac{\Omega+\Delta}{2\Omega}} \quad cos\theta_n = \sqrt{\tfrac{\Omega-\Delta}{2\Omega}} \qquad \Delta < 0 \end{aligned} \tag{4.5}$$

and arrive at the so called dressed states equations. Note that ω_R and thus Ω depend on n, which again makes θ_n dependent on n !

$$\begin{aligned} |+,n\rangle &= cos\theta_n|e,n\rangle + sin\theta_n|g,n+1\rangle \\ |-,n\rangle &= -sin\theta_n|e,n\rangle + cos\theta_n|g,n+1\rangle, \end{aligned} \tag{4.6}$$

which can be inverted to obtain

$$\begin{aligned} |e,n\rangle &= cos\theta_n|+,n\rangle - sin\theta_n|-,n\rangle \\ |g,n+1\rangle &= sin\theta_n|+,n\rangle + cos\theta_n|-,n\rangle. \end{aligned} \tag{4.7}$$

Irrespective of the sign of Δ it follows that:

$$\begin{aligned} sin2\theta_n = 2sin\theta_n cos\theta_n = \frac{\omega_R}{\Omega} \\ cos2\theta_n = cos^2\theta_n - sin^2\theta_n = -\frac{\Delta}{\Omega}. \end{aligned} \tag{4.8}$$

Note further that for $\frac{g_0}{\Delta} \to 0_+$ it follows that $|+\rangle \to |g,1\rangle$ and $|-\rangle \to |e,0\rangle$, whereas for $\frac{g_0}{\Delta} \to 0_-$ it is the other way round. This is obvious as for weak coupling and $\omega_c > \omega_{10}$ the upper dressed level $|+,n\rangle$ must naturally consist to the biggest part of the ground state $|g,n+1\rangle$ and vice versa.

4.2 Master equation in terms of dressed states

Let us express all the operators appearing in the master equation in terms of dressed states projectors, i.e. we set

$$\sigma^+ \equiv \sigma^+ \otimes \mathcal{E}_{Field} = |e\rangle\langle g| \otimes \sum_{n=0}^{\infty} |n\rangle\langle n|$$

$$\sigma^- \equiv \sigma^- \otimes \mathcal{E}_{Field} = |g\rangle\langle e| \otimes \sum_{n=0}^{\infty} |n\rangle\langle n|$$

$$\sigma^+\sigma^- \equiv \sigma^+\sigma^- \otimes \mathcal{E}_{Field} = |e\rangle\langle e| \otimes \sum_{n=0}^{\infty} |n\rangle\langle n|$$

$$a^+ \equiv \mathcal{E}_{atom} \otimes a^+ = (|e\rangle\langle e| + |g\rangle\langle g|) \otimes \sum_{n=0}^{\infty} \sqrt{n+1}|n+1\rangle\langle n|$$

$$a \equiv \mathcal{E}_{atom} \otimes a = (|e\rangle\langle e| + |g\rangle\langle g|) \otimes \sum_{n=0}^{\infty} \sqrt{n+1}|n\rangle\langle n+1|$$

$$a^+a \equiv \mathcal{E}_{atom} \otimes a^+a = (|e\rangle\langle e| + |g\rangle\langle g|) \otimes \sum_{n=0}^{\infty} n|n\rangle\langle n|$$

$$\mathcal{E}_{atom+field} = \mathcal{E}_{atom} \otimes \mathcal{E}_{field} = \sum_{n=0}^{\infty} |e,n\rangle\langle e,n| + |g,n\rangle\langle g,n|,$$

where $\mathcal{E}_{Field}, \mathcal{E}_{atom}$ and $\mathcal{E}_{atom+field}$ are the field, atom and combined atom-field system unity operators. The latter can be expressed in the dressed basis by the sum over the projectors onto the different manifolds + the projector onto the ground state:

$$\mathcal{E}_{atom+field} = |g,0\rangle\langle g,0| + \sum_{n=0}^{\infty} (\mathcal{P}_{+,n} + \mathcal{P}_{-,n}) \tag{4.9}$$

with

$$\mathcal{P}_{+,n} = |+,n\rangle\langle +,n| \qquad \mathcal{P}_{-,n} = |-,n\rangle\langle -,n|. \tag{4.10}$$

Using eqns. (4.7) the operators above can be entirely rewritten in a dressed basis, which yields rather unhandy expressions and just some examples are listed:

$$H_{J.C.} = \sum_{n=0}^{\infty} \lambda_n^+ \mathcal{P}_{+,n} + \lambda_n^- \mathcal{P}_{-,n} \tag{4.11}$$

$$\begin{aligned} \sigma^+\sigma^- = & \sum_{n=0}^{\infty} cos^2\theta_n \mathcal{P}_{+,n} + sin^2\theta_n \mathcal{P}_{-,n} \\ & -sin\theta_n cos\theta_n \left(|-,n\rangle\langle +,n| + |+,n\rangle\langle -,n|\right). \end{aligned} \tag{4.12}$$

As one can see, there are infinitely many coupled equations. So far no approximations have been made, either about the strength or the coherence properties of the cavity standing wave. In our case we make some assumptions below and proceed in a different, easier way to obtain equations for the lowest energy state $|g,0\rangle$ and the first pair of dressed states $|+,0\rangle, |-,0\rangle$ forming the first manifold.

4.2.1 The 3-state approximation

As before we assume that the pump beam is weak and that there is only one-photon at most in the standing wave inside the cavity. Without the pump beam the system decays to the state $|g,0\rangle$ and there is no occupation of higher levels in the dressed atom ladder. Due to the low intensity of the pump beam driving the cavity we assume that the system is excited only to the first dressed atom manifold consisting of $|+,0\rangle$ and $[-,0\rangle$, linear combinations of $|e,0\rangle$ and $|g,n+1\rangle$. Thus we only consider the three states

$$|g,0\rangle \equiv |0\rangle, |+,0\rangle \equiv |+\rangle, |-,0\rangle \equiv |-\rangle$$

and their respective couplings. They form a three dimensional state space.

It is easy to show that σ^+ only couples to a higher manifold and σ^- only couples to a lower manifold. Similarly, a^+ only couples a higher manifold and a only couples to a lower manifold, i.e.:

$$\begin{array}{lll} \langle i,m|\sigma^+|j,n\rangle = 0 & \langle i,m|a^+|j,n\rangle = 0 & m \neq n+1 \\ \langle i,m|\sigma^-|j,n\rangle = 0 & \langle i,m|a|j,n\rangle = 0 & m \neq n-1 \\ \langle i,m|\sigma^+\sigma^-|j,m\rangle = 0 & \langle i,m|a^+a|j,n\rangle = 0 & m \neq n, \end{array} \tag{4.13}$$

where $|i,m\rangle$ stands for either $|+,n\rangle$ or $|-,n\rangle$, the two states forming a manifold. Likewise, also $|g,0\rangle$ can be regarded as a one dimensional manifold at the bottom of the dressed states ladder and (4.13) may be applied. In appendix A the calculation of all nonzero operator matrix elements in the dressed basis is explicitly given (B.1). Using the master equation (2.8) and (4.13,B.1) it is easy to establish the equations of evolution for the system matrix elements. To do so, one has to insert the unity operator 4.9 in between the operator products and use eqn.(4.13) to retain only the non zero operator matrix elements. The remaining ones can be found through eqns.(B.1). The calculation of the essential matrix elements is given in the appendix B. (Note that $|\pm\rangle\langle\pm|$ is short for $|+\rangle\langle+| + |-\rangle\langle-|$, the unity operator within the $\mathcal{E}_0$ manifold. The same applies to $|\pm,1\rangle\langle\pm,1|$, the unity operator within the $\mathcal{E}_1$ manifold.) Those terms may thus be omitted according to our assumptions.

Putting the above results together, one finally obtains the following equations for the density matrix elements:

$$\begin{aligned}
\dot\rho_{0+} &= i\lambda_+\rho_{0+} - \Gamma\left(\rho_{0+}cos^2\theta - \rho_{0-}sin\theta cos\theta\right) - \kappa\left(\rho_{0+}sin^2\theta + \rho_{0-}cos\theta sin\theta\right) \\
&\quad -\eta\left(\rho_{++}sin\theta + \rho_{-+}cos\theta - \rho_{00}sin\theta\right) \\
\dot\rho_{0-} &= i\lambda_-\rho_{0-} - \Gamma\left(\rho_{0-}sin^2\theta - \rho_{0+}sin\theta cos\theta\right) - \kappa\left(\rho_{0-}cos^2\theta + \rho_{0+}cos\theta sin\theta\right) \\
&\quad -\eta\left(\rho_{--}cos\theta + \rho_{+-}sin\theta - \rho_{00}cos\theta\right) \\
\dot\rho_{00} &= 2\Gamma\left(\rho_{++}cos^2\theta + \rho_{--}sin^2\theta - \left(\rho_{-+} + \rho_{+-}\right)sin\theta cos\theta\right) \\
&\quad +2\kappa\left(\rho_{++}sin^2\theta + \rho_{--}cos^2\theta + \left(\rho_{+-} + \rho_{-+}\right)sin\theta cos\theta\right) \\
&\quad -\eta\left(\rho_{+0}sin\theta + \rho_{-0}cos\theta\right) - \eta\left(\rho_{0+}sin\theta + \rho_{0-}cos\theta\right) \\
\dot\rho_{++} &= -\Gamma\left(\rho_{++}2cos^2\theta - \left(\rho_{-+} + \rho_{+-}\right)sin\theta cos\theta\right) \\
&\quad -\kappa\left(\rho_{++}2sin^2\theta + \left(\rho_{-+} + \rho_{+-}\right)sin\theta cos\theta\right) + \eta\rho_{+0}sin\theta + \eta\rho_{0+}sin\theta \\
\dot\rho_{--} &= -\Gamma\left(\rho_{--}2sin^2\theta - \left(\rho_{-+} + \rho_{+-}\right)sin\theta cos\theta\right) \\
&\quad -\kappa\left(\rho_{--}2cos^2\theta + \left(\rho_{-+} + \rho_{+-}\right)sin\theta cos\theta\right) + \eta\rho_{-0}cos\theta + \eta\rho_{0-}cos\theta \\
\dot\rho_{-+} &= i\Omega\rho_{-+} - \Gamma\left(\rho_{-+} - \left(\rho_{++} + \rho_{--}\right)sin\theta cos\theta\right) \\
&\quad -\kappa\left(\rho_{-+} + \left(\rho_{++} + \rho_{--}\right)sin\theta cos\theta\right) + \eta\rho_{-0}sin\theta + \eta\rho_{0+}cos\theta \\
\dot\rho_{+-} &= \dot\rho^*_{-+}.
\end{aligned} \tag{4.14}$$

As one can see, $\dot{\rho}_{00}, \dot{\rho}_{--}, \dot{\rho}_{++}$ sum up to zero as expected.

4.2.2 Time evolution

The basic equation for ρ can be written like:

$$\left\{ \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \right\} \rho = \mathcal{L}\rho \tag{4.15}$$

As we assume to be in steady state, the first term on the left can be omitted. The remaining equation can be written in any chosen basis. Using the ordinary, uncoupled basis $|e,0\rangle, |g,1\rangle, |g,0\rangle$ and picking out the equation for $|g,1\rangle$, this gives :

$$v\langle g,1|\frac{\partial}{\partial x}\rho|g,1\rangle = v\frac{\partial}{\partial x}\langle g,1|\rho|g,1\rangle = \langle g,1|\mathcal{L}\rho|g,1\rangle. \tag{4.16}$$

Note that the partial derivative $\left(\frac{\partial}{\partial x}\right)$ commutes with $|g,1\rangle$, which does not explicitly depend on x. However, expressing eqn.4.15 in terms of the dressed basis introduced above, the states $|+\rangle$, $|-\rangle$ explicitly depend on x and do not commute with $\left(\frac{\partial}{\partial x}\right)$. Thus, their respective derivative must be taken into account as well, when expressing the left side of (4.15) in terms of dressed states.

Picking out the state $|+\rangle$, this gives:

$$\begin{aligned} v\langle +|\frac{\partial}{\partial x}\rho|+\rangle = & \\ & v\frac{\partial}{\partial x}\langle +|\rho|+\rangle - \left(\frac{\partial}{\partial x}\langle +|\right)|+\rangle - \langle +|\left(\frac{\partial}{\partial x}|+\rangle\right) \\ & = \langle +|\mathcal{L}\rho|+\rangle. \end{aligned}$$

Using the expression (4.6) for the dressed states, one can show — using the abbreviation $|\dot{+}\rangle \equiv v\frac{\partial}{\partial x}|+\rangle$— that:

$$\begin{aligned} \langle 0|\rho|\dot{+}\rangle + \langle \dot{0}|\rho|+\rangle &= v\nabla\theta\rho_{0-} \\ \langle 0|\rho|\dot{-}\rangle + \langle \dot{0}|\rho|-\rangle &= -v\nabla\theta\rho_{0+} \\ \langle \dot{+}|\rho|+\rangle + \langle +|\rho|\dot{+}\rangle &= v\nabla\theta\left(\rho_{-+} + \rho_{+-}\right) \\ \langle \dot{-}|\rho|-\rangle + \langle -|\rho|\dot{-}\rangle &= -v\nabla\theta\left(\rho_{-+} + \rho_{+-}\right) \\ \langle \dot{+}|\rho|-\rangle + \langle +|\rho|\dot{-}\rangle &= -v\nabla\theta\left(\rho_{++} - \rho_{--}\right) \\ \langle \dot{-}|\rho|+\rangle + \langle -|\rho|\dot{+}\rangle &= -v\nabla\theta\left(\rho_{++} - \rho_{--}\right). \end{aligned}$$

Above results can be used to express the left side of (4.15) in the dressed state basis and to specify finally

$$\begin{aligned} \dot{\rho}_{0+} &:= v\nabla\rho_{0+} - v\nabla\theta\rho_{0-} \\ \dot{\rho}_{++} &:= v\nabla\rho_{++} - v\nabla\theta\left(\rho_{-+} + \rho_{+-}\right) \end{aligned} \tag{4.17}$$

etc. in eqns.(4.14). As $Tr\{\rho\} = 1$, one can expres ρ_{00} in terms of ρ_{--}, ρ_{++}.

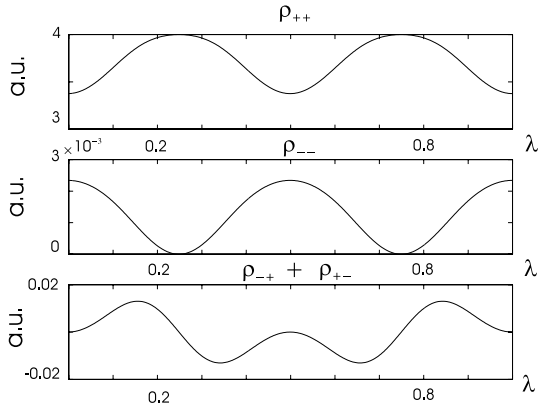

Figure 4.1: For $\Gamma = \kappa, g_0 = 2\kappa, \Delta = 6\kappa$ and $\Delta_c = 0$, $|+\rangle$ is strongly populated whereas ρ_{--} remains small.

4.3 Stationary population of the dressed levels

In the following we assume that the splitting Δ between the dressed levels $|+\rangle, |-\rangle$ is sufficiently large. We will show that in this limit the picture of an atom moving in a potential is valid. Note that this section is the continuation of section 3.2.1. Let us then look at two different situations.

1.) We tune our laser so that it is on resonance with the dressed level which turns into $|g, 1\rangle$ for zero coupling ($g = 0$). From the expressions obtained for the steady state density matrix it can be shown that it is almost exclusively that level which is populated. Furthermore, the coherences between the two dressed levels become negligibly small. This can be seen clearly in fig. 4.1, where $\Delta > 0$ and the upper level ρ_{++} corresponds to the unperturbed $|g, 1\rangle$. Its population is three orders of magnitude higher than that of the lower level.

2.) If, however, the laser is tuned to the dressed level, which turns into $|e, 0\rangle$ for zero coupling ($g = 0$), this is not the case at all. In fact, in that case the populations of both dressed states are about equal and very small compared to the populations obtained in 1.) Again fig. 4.2 shows that even when ρ_{--}, corresponding to the unperturbed $|e, 0\rangle$, is pumped resonantly, its population remains small and just about the same magnitude as ρ_{++}.

This is because the driving laser pumps the cavity and not the atom. It is only through the atom-cavity coupling g, that the atom 'sees' the pump laser. Driving the dressed state which is to the biggest part made up of $|g, 1 > -$ as was assumed in 1.) $-$ is therefore much more efficient. The atom is then effectively pumped into this state and can be thought of experiencing the potential and

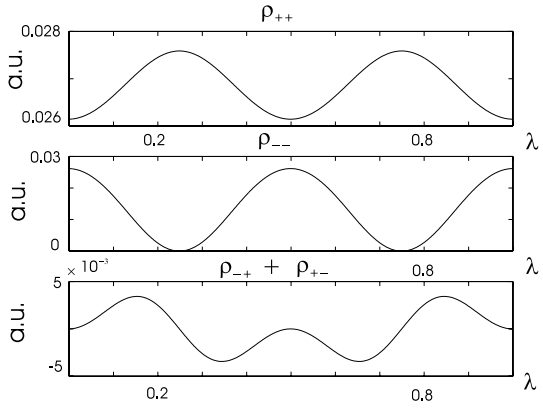

Figure 4.2: same parameters as in fig 4.1 apart from $\Delta_a = 0$ and $\Delta_c = -6$. Though ρ_{--} is pumped resonantly it remains hardly populated.

force connected to this level alone.

4.4 Comparison of 'Adiabatic elimination' and 'Dressed states' model

Coming back to (4.5),(4.3) we will now develop the expressions for $\Omega, sin^2\theta, cos^2\theta$ in powers of g^2/Δ_a^2 assuming that the cavity is driven near resonance i.e., $\Delta_c \approx 0, \Delta_a \approx \Delta$. In the following we set $\Delta > 0$, i.e. $\omega_c > \omega_{10}$ and we excite predominantly $|+\rangle$, the level that turns into $|g,1\rangle$ for zero coupling ($g = 0$).[1]

We obtain

$$\begin{aligned} \Omega &= \sqrt{\Delta^2 + 4g^2} \approx \Delta_a + \frac{2\Delta_a g^2}{\Delta_a^2} \\ sin^2\theta &= \frac{\Omega + \Delta}{2\Omega} \approx 1 \\ cos^2\theta &= \frac{\Omega - \Delta}{2\Omega} \approx \frac{g^2}{\Delta_a^2}. \end{aligned} \tag{4.18}$$

Now we try to find the force acting on the atom. Considering expression (B.17), we omit the coherence term (which is assumed to be small) and retain only the term proportional to ρ_{++} to obtain.

[1] For $\Delta < 0$ we would in turn have to pump $|-\rangle$

$$f \approx -\frac{\hbar\nabla\Omega}{2}\rho_{++} \approx -\frac{\nabla g^2\Delta_a}{\Delta_a^2}\rho_{++} \approx -\frac{\nabla g^2\Delta_a}{\Gamma^2+\Delta_a^2}\rho_{++}, \tag{4.19}$$

where we made the approximations

$$\Delta_a \gg g, \Gamma \tag{4.20}$$

and used (4.18). Those approximations correspond to the assumption that the dressed levels are clearly separated and their minimum separation Δ is much larger than g and Γ so that it is possible to excite only one level. Now we revert to (4.14) and note that the equations for ρ_{++} and ρ_{0+} can equivalently be written as follows:

$$\begin{aligned}
\dot{\rho}_{0+} &= i\lambda_+\rho_{0+} - \Gamma\left(\rho_{0+}cos^2\theta - \rho_{0-}sin\theta cos\theta\right) \\
&\quad -\kappa\left(\rho_{0+}sin^2\theta + \rho_{0-}cos\theta sin\theta\right) + \eta \\
\dot{\rho}_{++} &= -\Gamma\left(\rho_{++}2cos^2\theta - \left(\rho_{-+}+\rho_{+-}\right)sin\theta cos\theta\right) \\
&\quad -\kappa\left(\rho_{++}2sin^2\theta + \left(\rho_{-+}+\rho_{+-}\right)sin\theta cos\theta\right) \\
&\quad +\eta\left(\rho_{+0}+\rho_{0+}\right).
\end{aligned} \tag{4.21}$$

Inserting approximations (4.18) and using expression (4.2) for λ_+ we finally obtain

$$\begin{aligned}
\frac{d}{dt}\rho_{0+} &= i\left(-\Delta_c + \frac{\Delta_a g^2}{\Delta_a^2}\right)\rho_{0+} - \Gamma\frac{g^2}{\Delta_a^2}\rho_{0+} - \kappa\rho_{0+} + \eta \\
\frac{d}{dt}\rho_{++} &= \left(-2\Gamma\frac{g^2}{\Delta_a^2} - 2\kappa\right)\rho_{++} + \eta\left(\rho_{0+}+\rho_{+0}\right).
\end{aligned} \tag{4.22}$$

Note that for the time derivatives on the left side of above equations the additional coherences appearing according to (4.17) were omitted, because they are small for resonant pumping. On the right sides populations and coherences of the other dressed level $|-\rangle$ were omitted as well in the same approximation as in eqn.(4.20). Substituting $\Delta_a^2 \to \Delta_a^2 + \Gamma^2$ in the denominator (approx. 4.20) we obtain the same equations as for $\langle a^+\rangle, \langle a^+a\rangle$ in the adiabatic model (3.24), where $\frac{d}{dt}\langle a^+a\rangle$ can be found by differentiating the product and inserting (3.24). Also the expression for the force obtained from the dressed state model (4.19) is equivalent to the expression obtained from the adiabatic model (3.25) . Summing up it may be said that in the limit of large detuning $\Delta_a \gg \Gamma, g, \Delta_c$, the equations for the adiabatic model can be rederived in the dressed states model. For the dressed model coherences between the dressed levels can in this limit be neglected and only one level is effectively populated for resonant pumping. It is therefore justified to present a picture of an atom moving along a dressed level experiencing a force proportional to $-\nabla U(x) = -\nabla\Omega(x)$ and

proportional to the population of the dressed level (4.19) as is suggested by the simple Sisyphus cooling picture 3.4.

Chapter 5

Diffusion and temperature

We already saw in chapter 1 that there is a heating process due to random momentum transfer counteracting the cooling effect of the friction force on the atom. This prohibits that the atom completely stops. In equilibrium the contributions of friction and heating cancel and an equilibrium temperature can be inferred from the knowledge of D and β (1.46). The arguments for obtaining an expression for D in chapter 1 were rather vague and the situation is quite different for the case of a standing wave inside a cavity. We will therefore give a more rigorous mathematical derivation of the diffusion constant in a a cavity standing wave by use of the Quantum Regression Theorem (QRT). The QRT states that operator two time averages $\langle a_\nu(t)a_\mu(t')\rangle$ obey the same equations as one time averages $\langle a_\nu(t)\rangle$. If one can write

$$\frac{d}{dt}\langle a_\nu\rangle = \Lambda_{\nu\mu}\langle a_\mu\rangle \tag{5.1}$$

it follows that

$$\frac{d}{dt}\langle a_\nu a_\lambda\rangle = \Lambda_{\nu\mu}\langle a_\mu a_\lambda\rangle. \tag{5.2}$$

Now we come back to the definition of the diffusion of chapter(1) and derive an equation for it involving force correlations [23]:

$$\begin{aligned} 2D &= \frac{d}{dt}\Delta P^2(t) \\ &= \frac{d}{dt}\left(\langle P^2\rangle - \langle P\rangle\langle P\rangle\right) \\ &= 2\mathcal{R}e\left\{\langle PF\rangle - \langle P\rangle\langle F\rangle\right\} \\ &= 2\mathcal{R}e\int_{-\infty}^{0} dt\left(\langle F(t)F(0)\rangle - \langle F(t)\rangle\langle F(0)\rangle\right). \end{aligned} \tag{5.3}$$

For quasistationary conditions the above correlations are independent from the starting point of integration and the integral can be written as:

$$D = \mathcal{R}e \int_0^\infty dt \left(\langle F(0)F(t) \rangle - \langle F(0) \rangle \langle F(t) \rangle \right). \tag{5.4}$$

From (2.23) and the abbreviations (2.20) we know that the dipole force operator can be written as

$$F(x) = -\hbar g(x) X_1 \tag{5.5}$$

and thus calculating the dipole diffusion amounts to calculating the two time averages of the operatorX_1 . But there is also a second contribution, which we omitted for the average force as this contribution averaged over the vacuum field yields zero. It is the atom-vacuum coupling V_{AV} and it is this contribution we will deal with at first. One has

$$f_{vac} = -\mathbf{d} \nabla \mathbf{E}_{free}. \tag{5.6}$$

Now we know that

$$\langle f_{vac} \rangle = 0 \tag{5.7}$$

and thus (5.4) reads

$$D_{SE} = \mathcal{R}e \int_0^\infty dt \langle \mathbf{d}(0) \nabla \mathbf{E}_{free}(0) \mathbf{d}(t) \nabla \mathbf{E}_{free}(t) \rangle. \tag{5.8}$$

Above we used the fact that the dipole operator $\mathbf{d}$ commutes [1] at all times with the free field gradient, which can be proved using (1.14) and the fact that $\nabla \mathbf{E}_{source} = 0$. The free field operator products contain annihilation and creation operators whose average over the vacuum field is only nonzero for terms in antinormal ordering. Considering the rotating wave approximation, which includes terms like $a_k \sigma^+, \sigma^- a_k^+$, one obtains for the integrand

$$\begin{aligned} \langle \sigma^+(0) \sigma^-(t) \vec{d} \vec{\nabla} \mathbf{E}^+_{free}(0) \vec{d} \vec{\nabla} \mathbf{E}^-_{free}(t) \rangle = \\ \langle \sigma^+(0) \sigma^-(t) \rangle \sum_{\Delta k} \left(\vec{d} \vec{\epsilon}_k \right)^2 |\vec{k}|^2 \frac{\hbar \omega_k}{2 \epsilon_0 V} e^{+i \omega_k t} = \\ \langle \sigma^+(0) \sigma^-(t) \rangle \hbar k^2 2 \Gamma \delta(t), \end{aligned} \tag{5.9}$$

where the last line has been found by taking the sum over the two polarization directions, converting the sum into an integral over frequency and including

[1] In all calculations the atom is assumed to be stationary at a position x.

the free space density of modes. Then the Markoff approximation was made assuming the factors appearing next to the exponential to be quasi constant over the range of integration so that they can be taken out of the integral, which then can be approximated by a δ function. Taking the characteristic dipole radiation pattern into account this result would have to be modified by a constant factor ($\frac{2}{5}$ for circularly polarized light for instance).

Substituting this result for the integrand back into (5.8) and using (2.21) for $\langle \sigma^+ \sigma^- \rangle$ at steady state, we finally obtain

$$D_{SE} = \hbar^2 k^2 \Gamma \frac{\eta^2 g^2}{|det(A)|^2}. \tag{5.10}$$

Now we return to the calculation of the fluctuations of the dipole force. We want to introduce the following abbreviations:

$$\langle a_\nu a_\mu \rangle - \langle a_\nu \rangle \langle a_\mu \rangle \quad = \langle \delta a_\nu \delta a_\mu \rangle \quad =: \langle a_\nu, a_\mu \rangle, \tag{5.11}$$

where $\delta a_\nu = a_\nu - \langle a_\nu \rangle$ is the fluctuation of the operator a_ν around its mean and $\langle a_\nu, a_\mu \rangle$ is called the operator covariance. The part of the diffusion due to the dipole fluctuations can be written as

$$D_{dp} = \hbar^2 (\nabla g)^2 \mathcal{R}e \int_0^\infty dt \langle \delta X_1(0) \delta X_1(t) \rangle. \tag{5.12}$$

One can now introduce the Laplace transform

$$\begin{aligned} L\{g(t)\} &= \int_0^\infty dt e^{-st} g(t) \\ &= \tilde{g}(s) \end{aligned}$$

with the property

$$L\left\{ \frac{d}{dt} g(t) \right\} = -g(0) + s\tilde{g}(s) \tag{5.13}$$

and write (5.12) in terms of the Laplace transform of the covariance:

$$D_{dp} = \hbar^2 (\nabla g)^2 \mathcal{R}e \left\{ L\{\delta X_1(0) \delta X_1(t)\} \right\}|_{s=0}. \tag{5.14}$$

So one only needs the real part of the Laplace transform evaluated for $s = 0$ for the X_1 operator covariance. It turns out to be very convenient to have introduced a real matrix $\mathbf{B}$ (2.20) with the force operator being one of its basis elements. This requires only one covariance to be calculated ($\langle X_1(0), X_1(t) \rangle$ and

enables one to take the real part from the initial conditions only (see below), which considerably simplifies analytical efforts. Laplace transforms are easy to calculate for linear differential equations and they also take care of the intial conditions (c.f. 5.13) in a simple way.

Starting from (2.19) and (2.12) we can derive the following covariance equations where the QRT is invoked to obtain equations for the two time averages:

$$\frac{d}{dt}\langle X_1(0), \vec{Y}(t)\rangle \quad = \quad \mathbf{A}\langle X_1(0), \vec{Y}(t)\rangle, \tag{5.15}$$

where

$$\langle X_1(0), \vec{Y}(t)\rangle = \begin{pmatrix} \langle X_1(0), a(t)\rangle \\ \langle X_1(0), \sigma^-(t)\rangle. \end{pmatrix} \tag{5.16}$$

By comparison with (2.12) one sees that $\vec{Z}_\eta$ does not appear in this equation. This is because the inhomogenous part is cancelled by the second component of $\langle X_1(0), \vec{Y}(t)\rangle$ which is $\langle X_1(0)\rangle\langle\vec{Y}(t)\rangle$. This can be seen as follows: For $\langle\vec{Y}(t)\rangle$ in steady state we find

$$\frac{d}{dt}\langle X_1(0)\rangle\langle\vec{Y}(t)\rangle = 0$$

and by comparison with (2.12)

$$\mathbf{A}\langle\vec{Y}\rangle\langle X_1(0)\rangle = -\vec{Z}_\eta,$$

which shows why the inhomogenity on the right side of (2.12) cancels. Similarly one finds

$$\frac{d}{dt}\langle X_1(0), \vec{X}(t)\rangle \quad = \quad \mathbf{B}\langle X_1(0), \vec{X}(t)\rangle + \eta\langle X_1(0), \vec{I}\rangle \tag{5.17}$$

with analogous meaning of $\langle X_1(0), \vec{X}(t)\rangle$. One can now take the Laplace transforms on both sides, set $s = 0$ on the spot and retain only the initial conditions for the covariances. Inverting the resulting matrix equation we obtain

$$L\left\{\langle X_1(0), \vec{X}(t)\rangle\right\}\Big|_{s=0} = -\mathbf{B}^{-1}\left[\langle X_1(0), \vec{X}(0)\rangle + \eta L\left\{\langle X_1(0), \vec{I}(t)\rangle\right\}\Big|_{s=0}\right] \tag{5.18}$$

and considering that $\vec{I}$ is a linear combination of $\vec{Y}, \vec{Y}^+$ (2.20), the Laplace transforms of the last two have to be found analogously from (5.15) which yields

$$L\left\{\langle X_1(0), \vec{Y}(t)\rangle\right\}\Big|_{s=0} = -\mathbf{A}^{-1}\langle X_1(0), \vec{Y}(0)\rangle. \tag{5.19}$$

So solving (5.19) and inserting the result into (5.18) produces the desired correlation as the first vector element. Note further that as **B** is real and we further want to take the real part of the solution of (5.18), we simply take the real part of the multiplying vector on the right side.

Now the initial conditions need to be calculated from which the two sets of equations (5.19),(5.18) can be successively solved. For example we have:

$$\begin{aligned}\langle X_1(0), X_1(0)\rangle &= \langle X_1(0)X_1(0)\rangle - \langle X_1(0)\rangle\langle X_1(0)\rangle \\ &= \langle \left(a^+\sigma^- + a\sigma^+\right)\left(a^+\sigma^- + a\sigma^+\right)\rangle - \langle X_1(0)\rangle\langle X_1(0)\rangle \\ &= \langle a^+a\sigma^+\sigma^-\rangle + \langle aa^+\sigma^+\sigma^-\rangle - \langle X_1(0)\rangle\langle X_1(0)\rangle \\ &\approx \langle \sigma^+\sigma^-\rangle + \langle a^+a\rangle. \end{aligned} \tag{5.20}$$

For the last line another approximation was introduced. Up to now we only assumed the cavity to be weakly driven or far away from atomic resonance which results in a low saturation of the atomic transition. This would also allow for several intracavity photons to be accurately described. Above we assumed now that there is only one intracavity photon at the most, which implies

$$\eta \ll \Gamma, \kappa. \tag{5.21}$$

The operator a is then simply given by $|0\rangle_{cc}\langle 1|$ and the state space is reduced to the three states $|g,0\rangle, |g,1\rangle, |e,0\rangle$ already introduced in the last chapter for the dressed state model. One can then check that the above approximation becomes exact apart from the term $-\langle X_1(0)\rangle\langle X_1(0)\rangle$, which was omitted as it scales as η^4 whereas the other terms scale as η^2, Actually, **an exact calculation of (5.20) yields the same result as obtained in the above 'one photon approximation'**. This involves the evaluation of expectation values like $\langle a^+a\sigma^-\sigma^+\rangle$ but the problem can be solved with some tricky algebra and is treated in appendix C. For the derivation of all other initial conditions needed in (5.18,5.19) we can proceed in an analogous way. This is rather involved but does not present any problem and will therefore be omitted. We insert the results and take the real parts . Picking out the first element of the solution vector to (5.18) we obtain

$$\mathcal{R}e\left\{L\left\{\langle X_1(0), X_1(t)\rangle\right\}\right\}|_{s=0} = \frac{\eta^2}{|det(A)|^2}\Gamma\left(1 + \frac{4\Delta_a g^2}{\Gamma}\frac{\Delta_c\Gamma + \Delta_a\kappa}{|det(A)|^2}\right). \tag{5.22}$$

This result can be substituted into (5.14) and we finally get

$$\begin{aligned} D_{tot} &= D_{SE} + D_{dp} \\ &= \hbar^2k^2g^2\frac{\eta^2\Gamma}{|det(A)|^2} \end{aligned}$$

$$+ \quad \hbar^2 (\nabla g)^2 \frac{\eta^2 \Gamma}{|det(A)|^2} \left(1 + \frac{4\Delta_a g^2}{\Gamma} \frac{\Delta_c \Gamma + \Delta_a \kappa}{|det(A)|^2}\right). \tag{5.23}$$

Apart from the term appearing in the brackets next to 1 in the expression for D_{dp} this expression for the total diffusion is equivalent to the expression found in (3.17) for the classical treatment of the cavity mode when the coupling g_0 is replaced by $g0 \to g_0|\langle a\rangle_0|$ as suggested by Doherty et al. They argued that this would yield to a good approximation the correct diffusion as the intracavity state could be considered as quasi coherent (see section 3.1.1). The expression derived above now shows that this is not generally correct. The quantum nature of the mode introduces a correction term. The correction term can be much larger than the remaining terms, if the cavity is driven near resonance (like in the Doherty paper) and the splitting between cavity and atomic resonance $\Delta \approx \Delta_a$ is large. This is just the case when we want to make effective use of the new 'cavity induced cooling' mechanism, which can be pictured by a dressed state model with $\Delta_a \gg g, \Delta_c \approx 0$ (sections 3.2.1,4.4). In the opposite case, when the atom is driven resonantly and the 'Doppler cooling' force is dominant, $\Delta_a \approx 0$ and the correction term disappears. This is shown in fig. 5.1 below. The wavelength averaged diffusion with and without the corrective term is plotted against the laser-atom detuning Δ_a for fixed Δ. For Δ_a small they nicely agree whereas they strongly diverge for $\Delta_a \approx \Delta$.

Fig. 5.2 shows the spatial variation of the diffusion for $\Delta_a = 12$ where maximal diffusion occurs in fig. 5.1.

5.1 Sub-Doppler cooling

Now we have all the the terms needed for finding the equilibrium temperature as given in section 1.4. There we found the Doppler limit for classical cooling to be

$$k_B T = \frac{D}{\beta} \approx \hbar\Gamma. \tag{5.24}$$

To investigate the temperature limits for cavity cooling we simply take some typical parameters where this cooling force is maximal, which can be easily deduced from the dressed states picture. Hence it is possible to drive the cavity exactly resonant for $\omega_c > \omega_{10}$ or to drive the cavity slightly off resonance by an amount given by $\Delta_c = -\Delta/2 - \sqrt{\Delta^2 + 4g_0^2}$ for $\omega_c < \omega_{10}$. In both cases the atom is preferentially pumped into a 'valley' and decays on the way to the top losing kinetic energy. Those paramters may be inserted into the expressions for the diffusion and friction coefficient respectively and averaged over a wavelength. The temperature may then be found through

$$k_B T = \frac{\overline{D}}{\overline{\beta}} \tag{5.25}$$

-

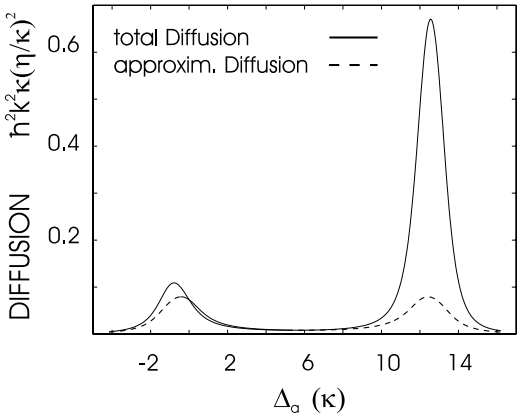

Figure 5.1: The total diffusion and the diffusion given by (3.17) plotted against the atomic detuning Δ_a for $\Delta = 12\kappa, \Gamma = \kappa, g_0 = 3\frac{1}{3}\kappa$.

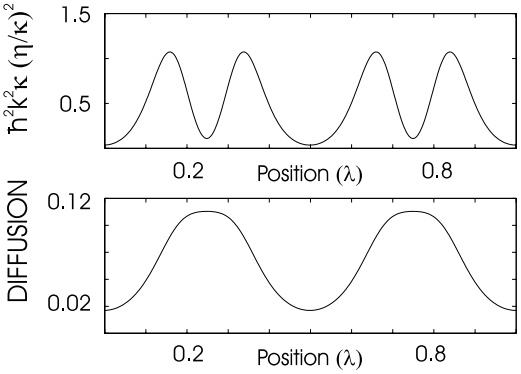

Figure 5.2: The total diffusion (above) and the approximated diffusion (below) for all parameters like in fig.5.1 and $\Delta_a = 12$ where maximal diffusion occurs.

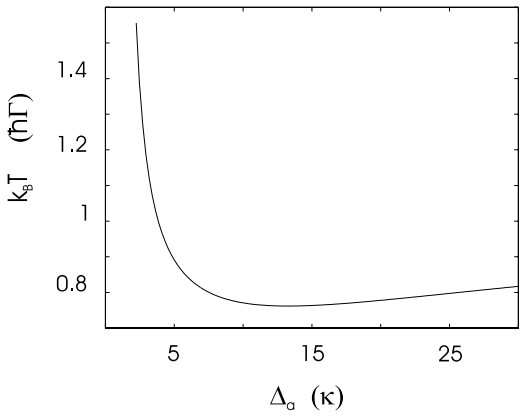

Figure 5.3: Temperature in units of the Doppler limit for pumping the node of the upper dressed level. $\Gamma = \kappa, g_0 = 3\frac{1}{3}\kappa, \Delta_c = -0.28\kappa$;

and plotted for a range of detunings. Fig.5.3 shows the variation of the temperature with Δ_a for $\Delta_c = -0.28\kappa$ and the other parameters as in fig.3.5. In this figure we are pumping the nodes of the upper level ($\Delta > 0, \Delta_c \approx 0$), whereas in fig.5.4, where $\Delta_c = -2\kappa$, we are pumping the antinodes of the lower level ($\Delta < 0, \Delta_c = -\Delta/2 - \sqrt{\Delta^2 + 4g_0^2}/2$). Compare with fig.3.6, where the friction force for those two areas is plotted.

One can see that the temperature, plotted in units of the Doppler limit $\hbar\Gamma$, can actually be slightly lower than the Doppler limit. Can this be possible? Is there a way to go far below the Doppler limit? One must consider that for Doppler cooling the energy transfer channel is the atomic spontaneous decay and temperatures in equilibrium can be as low as the width of this transfer channel (natural linewidth). In our case of 'cavity cooling', the dominant transfer channel is over the cavity and temperatures are determined by the width of the cavity decay rate. So choosing $\kappa \ll \Gamma$ should yield temperatures below the Doppler limit. If favorable parameters are chosen, the temperature may approach the cavity decay width which is much smaller than the natural linewidth. This is really what one finds and what is shown in fig.5.5, where the equilibrium temperature is plotted against the cavity loss rate κ. One clearly sees the linear dependence on κ for small κ, when the Sisyphus cooling is dominant. For κ larger than the height of the potential wells, which is determined by g and Δ, Sisyphus cooling disappears and Doppler cooling is dominant. The temperature therefore approaches the Doppler limit. In fig.5.5 it is actually slightly larger, as the detunings are not chosen such that maximal Doppler cooling occurs and consequently the Doppler limit cannot be exactly reached. Quantum Monte

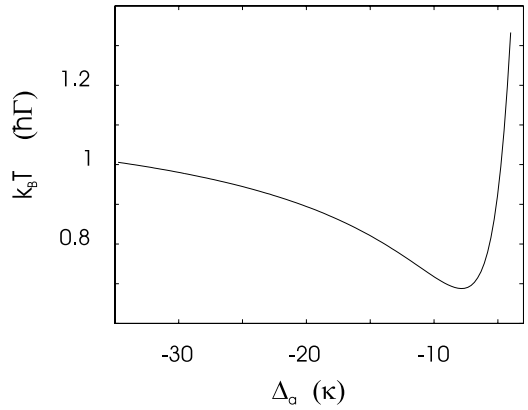

Figure 5.4: Temperature in units of the Doppler limit for pumping the antinode of the lower dressed level. $\Gamma = \kappa, g_0 = 3\frac{1}{3}\kappa, \Delta_c = -2\kappa$;

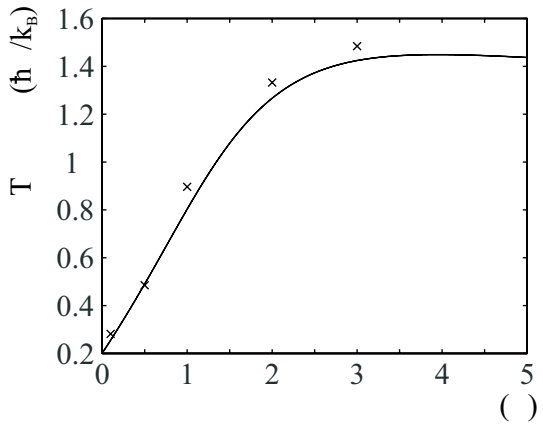

Figure 5.5: Going beyond the Doppler limit for $\Gamma \gg \kappa$. $g_0 = 2\Gamma, \Delta = -1.9\Gamma, \Delta_c = -1.3\Gamma$; The crosses mark the results obtained for Quantum Monte Carlo simulations.

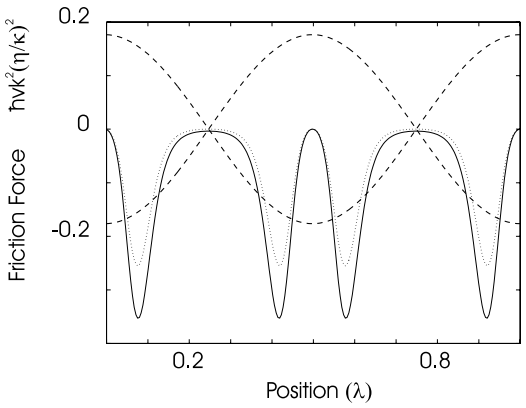

Figure 5.6: The spatial variation of the total friction force and the cavity mediated Sisyphus force f_{ca} (dotted line). The total force is slightly larger due to the damping contribution of the Doppler force. $\Gamma = 8\kappa$, other parameters as in fig.5.5.

Carlo simulations, the results of which are denoted by the black crosses along the curve, confirm the semiclasscially derived results. They were done by Peter Horak in a fully quantum (external degrees of freeedom quantized) model.

For fig.5.5 we have chosen $\Delta < 0$ and pumping the lower dressed level like shown in fig.3.4. This has the advantage that $\Delta_a < 0$ as well and so Doppler cooling adds additionally to the Sisyphus cooling effect. Fig.5.6 shows that the total friction force is slightly larger than the cavity mediated Sisyphus force alone. For higher atomic velocities

$$\frac{kv}{\kappa} \geq 1 \tag{5.26}$$

the Sisyphus effect becomes smaller like shown in fig.3.10. As $\Gamma \gg \kappa$, Doppler cooling is still very efficient for higher velocities (it reaches a maximum around $\frac{kv}{\Gamma} = \frac{1}{2}$) and thus becomes the dominant contribution to the force. It slows the faster atoms down until their velocities are small enough for Sisyphus cooling to appear, which in the low velocity limit in turn becomes the dominant contribution.

If we had chosen $\Delta > 0$ and pumped the node of the upper dressed level like shown in fig.3.3, the Doppler effect would have counteracted the Sisyphus cooling, which is shown in fig.3.8, where the total force is smaller than the cavity mediated force alone. For velocities satisfying eqn.(5.26) the antidamping Doppler effect becomes dominant and accelerates the atoms instead of cooling

them. This was also confirmed by the results of quantum Monte Carlo simulations done by Peter Horak, where cooling with above parameters appeared to be strongly dependent on the intital conditions chosen and only for very small initial momentum Doppler heating could be prevented and the atoms cooled.

5.2 Thermal distribution - Trapping conditions

For keeping an atom stably trapped it is necessary to keep the ratio

$$R = \frac{U_0}{E_{eq}} \tag{5.27}$$

as large a possible. Here U_0 indicates the depth of the potential well as given by (2.25) and E_{eq} the equilibrium temperature. Note that this potential pot is located at an antiode (fig.2.4) for $\Delta < 0$, when the lower dressed level is pumped resonantly as indicated in fig.3.4 and that the friction force is a minimum at the antinodes (fig.5.6). The diffusion, which is for the parameters of interest essentially only the corrective term, is largest between the node and the antinode (fig.5.2). An optical trap by definition consists of an optical field configuration with a point of stable equilibrium. Displacements of the atom result in a restoring force. If the equilibrium atomic kinetic energy is higher than the well depth, the atom can escape confinement. For standard Doppler cooling of chapter 1 the equilibrium energy can be as low as $\hbar\Gamma$ for certain parameters, but unfortunately the trap depth is found to be of the same magnitude. This does not allow for stable trapping to occur. For the new cooling scheme, however, higher Boltzmann factors may be obtained.

As an example: $\Gamma = 8\kappa, \Delta_a = -25.6\kappa, \Delta_c = -10.4\kappa, g_0 = 16\kappa$ yields $E_{eq} \approx 0.3\hbar\Gamma \approx 2.5\hbar\kappa$ and $U_0 \approx 0.25\hbar\kappa \left(\frac{\eta}{\kappa}\right)^2$. We see that the trap depth depends on the driving amplitude η. For sufficiently low saturation of the atomic transition we must choose a driving amplitude not larger than $\eta \approx 7\kappa$. Thus we arrive at a Boltzmann factor

$$R \approx \frac{0.25}{2.5} 50 = 5,$$

which is better than for standard Doppler cooling. Quantum Monte Carlo wave function simulations done by Peter Horak also find good localisation of the atom at the antinodes for similar parameters. This is an indication of the potential to stably trap a single atom in a cavity.

Chapter 6

Conclusions and comparison

In this work I derived an expression for the friction force of an atom moving in a cavity standing wave mode. This friction force can be split into the sum of a contribution related to the well known Doppler cooling and a new contribution arising from the cavity dynamics. The new part can be explained by a Sisyphus cooling scheme and is for certain parameters the dominant contribution. For the same parameters also a new term in the diffusion arises which partially counteracts the cooling process. For this new cooling scheme the cavity is driven more or less at resonance and the dominant decay channel is through the cavity κ, whereas for Doppler cooling the atom is pumped near resonance and the decay is through spontaneous emission. In the Sisyphus cooling picture of an atom in a weakly driven cavity it is either possible to pump the atom at the antinodes of the lower dressed level for $\omega_c < \omega_{10}$ or to pump the atom at the nodes of the upper dressed level for $\omega_c > \omega_{10}$. For $\kappa \ll \Gamma$ the Doppler limit of the atomic temperature may be surpassed as the atomic equilibrium temperature is limited by the cavity decay width κ, just as it approaches the natural linewidth Γ for Doppler cooling. This may be used to actually cool a two level system below the Doppler limit inside a cavity mode field. Also trapping of an atom seems feasible as Boltzmann factors larger than unity may be gained. All the results derived for the temperature are in very good agreement to fully quantum Monte Carlo wave function simulations in the semiclassical limit. Note that the calculations still need to be generalised to the three dimensional case. Using a matrix continued fraction method as outlined in [22] it would be possible to calculate the total cooling force for arbitrary velocities. As the new cavity induced cooling effects are based on the experimental setups used by Kimble et al. [9], the results should have major impact on ongoing experiments with laser cooled atoms in high finesse cavities.

Appendix A

Dipole force

A.1 Expression for the friction force

The total friction force can be regarded as the sum of the 'Doppler cooling' force f_{at} (obtained through adiabatic elimination of the mode) and the 'cavity cooling' force f_{ca}(obtained through adiabatic elimination of the atom):

$$f_1 = f_{at} + f_{ca}, \tag{A.1}$$

where

$$\begin{aligned} f_{at} \;=\; & -\hbar(\nabla g)^2\eta^2 4(-\Delta_a^3\Delta_c^4\Gamma - \Delta_a^2\Delta_c^3 g^2\Gamma + \Delta_a\Delta_c^2 g^4\Gamma + \Delta_c g^6\Gamma - \Delta_a\Delta_c^4\Gamma^3 + \Delta_c^3 g^2\Gamma^3 \\ & -2\Delta_a^3\Delta_c^2 g^2\kappa + 2\Delta_a g^6\kappa + 2\Delta_c g^4\Gamma^2\kappa - 2\Delta_a^3\Delta_c^2\Gamma\kappa^2 - \Delta_a^2\Delta_c g^2\Gamma\kappa^2 + 3\Delta_a g^4\Gamma\kappa^2 \\ & -2\Delta_a\Delta_c^2\Gamma^3\kappa^2 + \Delta_c g^2\Gamma^3\kappa^2 - 2\Delta_a^3 g^2\kappa^3 - \Delta_a^3\Gamma\kappa^4 - \Delta_a\Gamma^3\kappa^4)\frac{1}{|det(A)|^6} \end{aligned} \tag{A.2}$$

$$\begin{aligned} f_{ca} \;=\; & -\hbar(\nabla g)^2\eta^2 4(-2\Delta_a^2\Delta_c^2\Gamma + 2g^4\Gamma - 2\Delta_c^2\Gamma^3 - 4\Delta_a^3\Delta_c\kappa + 4\Delta_a^2 g^2\kappa \\ & -4\Delta_a\Delta_c\Gamma^2\kappa + 4g^2\Gamma^2\kappa + 2\Delta_a^2\Gamma\kappa^2 + 2\Gamma^3\kappa^2)\frac{\Delta_a g^2}{|det(A)|^6} \end{aligned} \tag{A.3}$$

and

$$|det(A)|^6 = \left[\left(\Gamma\kappa + g^2 - \Delta_a\Delta_c\right)^2 + (\Delta_a\kappa + \Delta_c\Gamma)^2\right]^3. \tag{A.4}$$

Appendix B

Dressed states

At first the tables of matrix elements of atom-mode operator products needed for the dressed representation of the master equation in chapter 3 are given. Then we continue from where we ended in chapter 3, that is we will calculate the force acting on the atom up to first order in v. Expressions for the force and the mean photon number will be derived which are the same as the ones already obtained in chapter 1.

B.1 Operator matrix elements

For obtaining all non-zero operator matrix elements in the dressed basis (4.13) and (4.6) may be applied. In our 3-dim. state space one finds for the operators $a, a^+, a^+a, \sigma^-, \sigma^+, \sigma^+\sigma-$:

$$\begin{aligned}
\langle 0|a|+\rangle &= \langle 0,g|a|\left(cos\theta|e,0\rangle + sin\theta|g,1\rangle\right) = sin\theta \\
\langle 0|a|-\rangle &= \langle 0,g|a|-,0\rangle = cos\theta \\
\langle +|a^+a|+\rangle &= sin^2\theta\langle g,1|a^+a|g,1\rangle = sin^2\theta \\
\langle -|a^+a|-\rangle &= cos^2\theta\langle g,1|a^+a|g,1\rangle = cos^2\theta \\
\langle -|a^+a|+\rangle &= sin\theta cos\theta\langle g,1|a^+a|g,1\rangle = sin\theta cos\theta \\
\langle 0|\sigma^-|+\rangle &= cos\theta\langle 0,g|\left(|g,0\rangle\langle e,0|\right)|e,0\rangle = cos\theta \\
\langle 0|\sigma^-|-\rangle &= -sin\theta\langle 0,g|\left(|g,0\rangle\langle e,0|\right)|e,0\rangle = -sin\theta \\
\langle +|\sigma^+\sigma^-|+\rangle &= cos^2\theta\langle e,0|\left(|e,0\rangle\langle e,0|\right)|e,0\rangle = cos^2\theta \\
\langle -|\sigma^+\sigma^-|-\rangle &= sin^2\theta\langle e,0|\left(|e,0\rangle\langle e,0|\right)|e,0\rangle = cos^2\theta \\
\langle +|\sigma^+\sigma^-|-\rangle &= -sin\theta cos\theta. \qquad \text{(B.1)}
\end{aligned}$$

B.2 Master equation represenation in $|+\rangle, |-\rangle, |0\rangle$

$\dot{\rho}_{0+}$:

$$
\begin{aligned}
\langle 0|H_{J.C.}\rho - \rho H_{J.C.}|+\rangle &= -\rho_{0+}\lambda_{+} \\
\langle 0|\sigma^{+}\sigma^{-}\rho|+\rangle &= 0 \\
\langle 0|\rho\sigma^{+}\sigma^{-}|+\rangle &= \langle 0|\rho|\pm\rangle\langle\pm|\sigma^{+}\sigma^{-}|\pm\rangle \\
&= \rho_{0+}cos^{2}\theta - \rho_{0-}sin\theta cos\theta \\
\langle 0|\sigma^{q} - \rho\sigma^{+}|+\rangle &= \langle 0|\sigma^{-}|\pm\rangle\langle\pm|\rho||\pm,1\rangle\langle\pm,1|\sigma^{+}|+\rangle \\
\langle 0|a^{+}a\rho|+\rangle &= 0 \\
\langle 0|\rho a^{+}a|+\rangle &= \langle 0|\rho|\pm\rangle\langle\pm|a^{+}a|+\rangle \\
&= \rho_{0+}sin^{2}\theta + \rho_{0-}cos\theta sin\theta \\
\langle 0|a\rho a^{+}|+\rangle &= \langle 0|a|\pm\rangle\langle\pm|\rho|\pm,1\rangle\langle\pm,1|a^{+}|+\rangle \\
\langle 0|a\rho|+\rangle &= \langle 0|a|\pm\rangle\langle\pm|\rho|+\rangle \\
&= sin\theta\rho_{++} + cos\theta\rho_{-+} \\
\langle 0|\rho a|+\rangle &= \langle 0|\rho|0\rangle\langle 0|a|+\rangle \\
&= \rho_{00}sin\theta \\
\langle 0|a^{+}\rho|+\rangle &= 0 \\
\langle 0|\rho a^{+}|+\rangle &= \langle 0|\rho|\pm,1\rangle\langle\pm,1|a^{+}|+\rangle
\end{aligned}
$$

$\dot{\rho}_{0-}$:

$$
\begin{aligned}
\langle 0|H_{J.C.}\rho - \rho H_{J.C.}|-\rangle &= -\rho_{0-}\lambda_{-} \\
\langle 0|\sigma^{+}\sigma^{-}\rho|-\rangle &= 0 \\
\langle 0|\rho\sigma^{+}\sigma^{-}|-\rangle &= \langle 0|\rho|\pm\rangle\langle\pm|\sigma^{+}\sigma^{-}|\pm\rangle \\
&= rho_{0-}cos^{2}\theta - \rho_{0+}sin\theta cos\theta \\
\langle o|\sigma^{-}\rho\sigma^{+}|-\rangle &= \langle 0|\sigma^{-}|\pm\rangle\langle\pm|\rho||\pm,1\rangle\langle\pm,1|\sigma^{+}|-\rangle \\
\langle 0|a^{+}a\rho|-\rangle &= 0 \\
\langle 0|\rho a^{+}a|-\rangle &= \langle 0|\rho|\pm\rangle\langle\pm|a^{+}a|-\rangle \\
&= \rho_{0-}cos^{2}\theta + \rho_{0+}cos\theta sin\theta \\
\langle 0|a\rho a^{+}|-\rangle &= \langle 0|a|\pm\rangle\langle\pm|\rho|\pm,1\rangle\langle\pm,1|a^{+}|-\rangle \\
\langle 0|a\rho|-\rangle &= \langle 0|a|\pm\rangle\langle\pm|\rho|-\rangle \\
&= cos\theta\rho_{--} + sin\theta\rho_{+-} \\
\langle 0|\rho a|-\rangle &= \langle 0|\rho|0\rangle\langle 0|a|-\rangle \\
&= \rho_{00}cos\theta \\
\langle 0|a^{+}\rho|-\rangle &= 0 \\
\langle 0|\rho a^{+}|-\rangle &= \langle 0|\rho|\pm,1\rangle\langle\pm,1|a^{+}|-\rangle
\end{aligned}
$$

$\dot{\rho}_{00}$:

$$
\begin{aligned}
\langle 0|H_{J.C.}\rho - \rho H_{J.C.}|0\rangle &= 0 \\
\langle 0|\sigma^+\sigma^-\rho|0\rangle &= 0 \\
\langle 0|\rho\sigma^+\sigma^-|0\rangle &= 0 \\
\langle o|\sigma^-\rho\sigma^+|0\rangle &= \langle 0|\sigma^-|\pm\rangle\langle\pm|\rho|\pm\rangle\langle\pm|\sigma^+|0\rangle \\
&= cos^2\theta\rho_{++} + sin^2\theta\rho_{--} - cos\theta sin\theta\,(rho_{-+} + \rho_{+-}) \\
\langle 0|a^+a\rho|0\rangle &= 0 \\
\langle 0|\rho a^+a|0\rangle &= 0 \\
\langle 0|a\rho a^+|0\rangle &= \langle 0|a|\pm\rangle\langle\pm|\rho|\pm\rangle\langle\pm|a^+|0\rangle \\
&= sin^2\theta\rho_{++} + cos^2\theta\rho_{--} + sin\theta cos\theta\,(\rho_{-+} + \rho_{+-}) \\
\langle 0|a\rho|0\rangle &= \langle 0|a|\pm\rangle\langle\pm|\rho|0\rangle \\
&= cos\theta\rho_{-0} + sin\theta\rho_{+0} \\
\langle 0|\rho a|0\rangle &= 0 \\
\langle 0|a^+\rho|0\rangle &= 0 \\
\langle 0|\rho a^+|0\rangle &= \langle 0|\rho|\pm\rangle\langle\pm|a^+|0\rangle \\
&= sin\theta\rho_{0+} + cos\theta\rho_{0-}
\end{aligned}
$$

$\dot{\rho}_{++}$:

$$
\begin{aligned}
\langle +|H_{J.C.}\rho - \rho H_{J.C.}|+\rangle &= 0 \\
\langle +|\sigma^+\sigma^-\rho|+\rangle &= \langle +|\sigma^+\sigma^-|\pm\rangle\langle\pm|\rho|+\rangle \\
&= \cos^2\theta\rho_{++} - sin\theta cos\theta\rho_{-+} \\
\langle +|\rho\sigma^+\sigma^-|+\rangle &= \langle +|\rho|\pm\rangle\langle\pm|\sigma^+\sigma^-|+\rangle \\
&= \rho_{++}cos^2\theta - \rho_{+-}sin\theta cos\theta \\
\langle +|\sigma^-\rho\sigma^+|+\rangle &= \langle +|\sigma^-|\pm,1\rangle\langle\pm,1|\rho||\pm,1\rangle\langle\pm,1|\sigma^+|+\rangle \\
\langle +|a^+a\rho|+\rangle &= \langle +|a^+a|\pm\rangle\langle\pm|\rho|+\rangle \\
&= \rho_{++}sin^2\theta + \rho_{-+}sin\theta cos\theta \\
\langle +|\rho a^+a|+\rangle &= \langle +|\rho|\pm\rangle\langle\pm|a^+a|+\rangle \\
&= \rho_{++}sin^2\theta + \rho_{+-}sin\theta cos\theta \\
\langle +|a\rho a^+|+\rangle &= \langle +|a|\pm,1\rangle\langle\pm,1|\rho|\pm,1\rangle\langle\pm,1|a^+|+\rangle \\
\langle +|a\rho|+\rangle &= \langle +|a|\pm,1\rangle\langle\pm,1|\rho|+\rangle \\
\langle +|\rho a|+\rangle &= \langle +|\rho|0\rangle\langle 0|a|+\rangle \\
&= \rho_{+0}sin\theta \\
\langle +|a^+\rho|+\rangle &= \langle +|a^+|0\rangle\langle 0|\rho|+\rangle \\
&= \rho_{0+}sin\theta \\
\langle +|\rho a^+|+\rangle &= \langle +|\rho|\pm,1\rangle\langle\pm,1|a^+|+\rangle
\end{aligned}
$$

$\dot{\rho}_{--}$:

$$
\begin{aligned}
\langle -|H_{J.C.}\rho - \rho H_{J.C.}|-\rangle &= 0 \\
\langle -|\sigma^+\sigma^-\rho|-\rangle &= \langle -|\sigma^+\sigma^-|\pm\rangle\langle\pm|\rho|-\rangle \\
&= \sin^2\theta\rho_{--} - sin\theta cos\theta\rho_{+-} \\
\langle -|\rho\sigma^+\sigma^-|-\rangle &= \langle -|\rho|\pm\rangle\langle\pm|\sigma^+\sigma^-|-\rangle \\
&= \rho_{--}sin^2\theta - \rho_{-+}sin\theta cos\theta \\
\langle -|\sigma^-\rho\sigma^+|-\rangle &= \langle -|\sigma^-|\pm,1\rangle\langle\pm,1|\rho||\pm,1\rangle\langle\pm,1|\sigma^+|-\rangle \\
\langle -|a^+a\rho|-\rangle &= \langle -|a^+a|\pm\rangle\langle\pm|\rho|-\rangle \\
&= \rho_{--}cos^2\theta + \rho_{+-}sin\theta cos\theta \\
\langle -|\rho a^+a|-\rangle &= \langle -|\rho|\pm\rangle\langle\pm|a^+a|-\rangle \\
&= \rho_{--}cos^2\theta + \rho_{-+}sin\theta cos\theta \\
\langle -|a\rho a^+|-\rangle &= \langle -|a|\pm,1\rangle\langle\pm,1|\rho|\pm,1\rangle\langle\pm,1|a^+|-\rangle \\
\langle -|a\rho|-\rangle &= \langle -|a|\pm,1\rangle\langle\pm,1|\rho|-\rangle \\
\langle -|\rho a|-\rangle &= \langle -|\rho|0\rangle\langle 0|a|-\rangle \\
&= \rho_{-0}cos\theta \\
\langle -|a^+\rho|-\rangle &= \langle -|a^+|0\rangle\langle 0|\rho|-\rangle \\
&= \rho_{0-}cos\theta \\
\langle -|\rho a^+|-\rangle &= \langle -|\rho|\pm,1\rangle\langle\pm,1|a^+|-\rangle
\end{aligned}
$$

$\dot{\rho}_{-+}$:

$$
\begin{aligned}
\langle -|H_{J.C.}\rho - \rho H_{J.C.}|+\rangle &= \lambda_-\rho_{-+} - \rho_{-+}\lambda_+ \\
&= -\Omega\rho_{-+} \\
\langle -|\sigma^+\sigma^-\rho|+\rangle &= \langle -|\sigma^+\sigma^-|\pm\rangle\langle\pm|\rho|+\rangle \\
&= \rho_{-+}sin^2\theta - sin\theta cos\theta\rho_{++} \\
\langle -|\rho\sigma^+\sigma^-|+\rangle &= \langle -|\rho|\pm\rangle\langle\pm|\sigma^+\sigma^-|+\rangle \\
&= \rho_{-+}cos^2\theta - \rho_{--}sin\theta cos\theta \\
\langle -|\sigma^-\rho\sigma^+|+\rangle &= \langle -|\sigma^-|\pm,1\rangle\langle\pm,1|\rho||\pm,1\rangle\langle\pm,1|\sigma^+|+\rangle \\
\langle -|a^+a\rho|+\rangle &= \langle -|a^+a|\pm\rangle\langle\pm|\rho|+\rangle \\
&= \rho_{-+}cos^2\theta + \rho_{++}sin\theta cos\theta \\
\langle -|\rho a^+a|+\rangle &= \langle -|\rho|\pm\rangle\langle\pm|a^+a|+\rangle \\
&= \rho_{-+}sin^2\theta + \rho_{--}sin\theta cos\theta \\
\langle -|a\rho a^+|+\rangle &= \langle -|a|\pm,1\rangle\langle\pm,1|\rho|\pm,1\rangle\langle\pm,1|a^+|+\rangle \\
\langle -|a\rho|+\rangle &= \langle -|a|\pm,1\rangle\langle\pm,1|\rho|+\rangle \\
\langle -|\rho a|+\rangle &= \langle -|\rho|0\rangle\langle 0|a|+\rangle = \rho_{-0}sin\theta \\
\langle -|a^+\rho|+\rangle &= \langle -|a^+|0\rangle\langle 0|\rho|+\rangle \\
&= \rho_{0+}cos\theta \\
\langle -|\rho a^+|+\rangle &= \langle -|\rho|\pm,1\rangle\langle\pm,1|a^+|+\rangle
\end{aligned}
$$

B.3 Expansion up to first order in v

In the case of an atom moving slowly in a cavity standing wave the solution of eqn.(2.8) can be expanded to first order in v. This expansion is valid only for $\frac{kv}{\Gamma} \ll 1$. This means that the atom doesn't move far on the scale of the cavity mode wavelength before it decays spontaneously. Terms of higher order than first order in v are neglected. Thus, with respect to eqn.(4.15)

$$\mathcal{L}\rho_0 = 0 \tag{B.2}$$

provides the zeroth order solution and

$$v\frac{\partial}{\partial x}\rho_0 = \mathcal{L}\rho_1 \tag{B.3}$$

provides the first order solution in v (c.f.(1.34)). The same we will do now with the equations expressed in dressed states together with a perturbation theory in the driving field amplitude η.

As outlined above, η is assumed to be very small to excite only the first manifold. Without any external driving field the atom is assumed to be in the ground state and no photon in the cavity. Even with a weak perturbation the occupation number of the ground state ρ_{00} will be close to 1, whereas the occupation numbers of the higher levels ρ_{++} and ρ_{--} will be of order of η^2. Eqns.(4.14) can be written in a compact form:

$$v\mathbf{D}\vec{\rho} \;\; = \;\; \mathbf{M}\vec{\rho} + \vec{K}_\eta, \tag{B.4}$$

where $\mathbf{D}$ stands for a matrix with elements combining linear differential operators(∇) with factors like $\nabla\theta$, as is given by eqn.4.17. $\mathbf{M}$ again can be split into two parts

$$\mathbf{M} = \mathbf{M}_o + \mathbf{M}_\eta, \tag{B.5}$$

where $\mathbf{M}_\eta$ stands for a matrix containing the driving amplitude η (to first order) which represents together with $\mathbf{M}_o$ the homogeneous part in $\vec{\rho}$, which in turn is a column vector:

$$\vec{\rho} = (\rho_{0+}, \rho_{0-}, \rho_{++}, \rho_{--}, \rho_{-+}, \rho_{+-})\,.$$

The column vector $\vec{K}_\eta$ represents the inhomogeneous part, which appeared when we set $\rho_{00} = 1 - \rho_{++} - \rho_{--}$ in eqns.(4.14).Note that all but the first two elements of $\vec{K}_\eta$ are zero. Solving eqn.(B.4) for $v = 0$:

$$\mathbf{M}\vec{\rho}_{(0)} = -\vec{K}_\eta. \tag{B.6}$$

At the same time we only want to retain terms of order of η^2 or less and thus we may drop the driving terms in eqns.(4.14) which are proportional to $\rho_{++}\eta, \rho_{--}\eta$. This results in a **decoupling** of the equations for ρ_{0+}, ρ_{0-} from the remaining ones. First the equations coupling only ρ_{0+} and ρ_{0-} are solved with the inhomogenity given by $\vec{K}_\eta$. The result is inserted into the driving terms (which are only governed by ρ_{0+}, ρ_{0-}) of the remaining equations, which again are solved for the resulting inhomogenity.

This procedure yields coherences ρ_{0+}, ρ_{0-} of first order in η and populations of (and coherences between) the higher levels which are of second order in η as was assumed. $\vec{\rho}_{(0)}$ (the solution for v=0) can be split into

$$\rho_{(0)} = \vec{\rho}_{(01)} + \vec{\rho}_{(02)}, \tag{B.7}$$

where $\rho_{(01)}$ is first order in η and $\rho_{(02)}$ is second order in η. For the solution to first order in v, one can use the result obtained for v=0. Keeping the terms first order in v, eqn.(B.4) can be written analogously to eqn.(1.34):

$$v\mathbf{D}\vec{\rho}_{(0)} = \mathbf{M}\vec{\rho}_{(1)}. \tag{B.8}$$

Here again we have to take into account the perturbation theorie in η, which allows us only to keep the terms up to η^2. Thus we expand $\vec{\rho}_{(1)}$ the same way as $\vec{\rho}_{(0)}$:

$$\vec{\rho}_{(1)} = \vec{\rho}_{(11)} + \vec{\rho}_{(12)}, \tag{B.9}$$

where the terms are first and second order in η respectively. This gives the following equations:

$$v\left(\vec{I}_{(1)} + \vec{I}_{(2)}\right) = (\mathbf{M}_o + \mathbf{M}_\eta)\left(\vec{\rho}_{(11)} + \vec{\rho}_{(12)}\right), \tag{B.10}$$

where

$$\begin{aligned} \vec{I}_{(1)} &= \mathbf{D}\vec{\rho}_{(01)} \\ \vec{I}_{(2)} &= \mathbf{D}\vec{\rho}_{(02)}. \end{aligned} \tag{B.11}$$

Note that $\mathbf{D}$ only combines elements of first order with other elements of first order in η. The same holds for elements second order in η. Thus, only the first two elements of $\vec{I}_{(1)}$ (terms in ρ_{0+}, ρ_{0-})are nonzero. For $\vec{I}_{(2)}$, the first two elements are zero, whereas the remaining ones (terms in $\rho_{++}, \rho_{--}, ...$) are not. Comparing terms of same order in η yields:

$$\begin{aligned} v\vec{I}_{(1)} &= \mathbf{M}_o\vec{\rho}_{(11)} \\ v\vec{I}_{(2)} &= \mathbf{M}_o\vec{\rho}_{(12)} + \mathbf{M}_\eta\vec{\rho}_{(11)} \end{aligned} \tag{B.12}$$

and so

$$\begin{aligned}\vec{\rho}_{(12)} &= \mathbf{M}_o^{-1}\left(v\vec{I}_{(2)} - \mathbf{M}_\eta\vec{\rho}_{11}\right) \\ &= v\mathbf{M}_o^{-1}\left(\vec{I}_{(2)} - \mathbf{M}_\eta\mathbf{M}_o^{-1}\vec{I}_{(1)}\right). \end{aligned} \tag{B.13}$$

This procedure for obtaining the solution for first order in v ($\vec{\rho}_{(1)}$) is analogous to the one used for $\vec{\rho}_{(0)}$. One has to solve a 2*2 equation set (right sides of the first two of eqns. (4.14) without driving terms) for $\{\rho_{0+}, \rho_{0-}\}$ with the inhomogenity this time being in principle the derivative of the zeroth order solution. This yields the terms of first order in η. The result is inserted into the inhomogenity of a 4*4 equation set (last four equations of eqns.(4.14) without driving terms) of $\{\rho_{++}, \rho_{--}, \rho_{-+}, \rho_{+-}\}$ where the second part of the inhomogenity is again in principle the derivative of the zeroth order solution . The 4*4 equation set then has to be solved to obtain the terms of second order in η.

Summarising, the matrix $\mathbf{M}_o$ defined above, which is in principle the matrix defining the last four equations of eqns.(4.14) without the driving terms on the right side, may be split into two submatrices (2*2 and 4*4) which have to be diagonalised for obtaining the zeroth order terms of ρ as well as the first order terms. In order to diagonalise these matrices more easily it is useful to take advantage of the symmetries found in their structure and define linear combinations of their basis variables and express them in terms of this newly found basis. It turns out to be convenient to introduce linear combinations

$$\begin{aligned}\rho_a &= \rho_{++} + \rho_{--} \\ \rho_b &= \rho_{++} - \rho_{--} \\ \rho_g &= \rho_{-+} + \rho_{+-} \\ \rho_u &= \rho_{-+} - \rho_{+-}.\end{aligned}$$

B.3.1 Two important matrices

Using eqns.(4.14), forming linear combinations and omitting the driving terms, using basic trigonometric theorems and eqns.(4.8), one finds the following expressions for the (2*2) and (4*4) matrix systems:

$$\mathbf{A}(\rho_{0+}, \rho_{0-}) = \begin{pmatrix} i\lambda_+ - \gamma/2 - \delta\gamma/2 & \gamma_{+-} \\ \gamma_{+-} & i\lambda_- - \gamma/2 + \delta\gamma/2 \end{pmatrix} \tag{B.14}$$

$$\mathbf{B}(\rho_a,\rho_b,\rho_g,\rho_u) = \begin{pmatrix} -\gamma & -\delta\gamma & \gamma_{+-} & 0 \\ -\delta\gamma & -\gamma & 0 & 0 \\ \gamma_{+-} & 0 & -\gamma & i\Omega \\ 0 & 0 & i\Omega & -\gamma \end{pmatrix}, \tag{B.15}$$

where

$$\begin{aligned} \gamma &= (\Gamma+\kappa) \\ \delta\gamma &= \frac{\Delta}{\Omega}(\kappa-\Gamma) \\ \gamma_{+-} &= \frac{\omega_R}{\Omega}(\Gamma-\kappa). \end{aligned}$$

The determinants of the two matrices are

$$\begin{aligned} det(A) &= -\lambda_+\lambda_- + i\Delta_c\gamma + i\Delta\kappa + \Gamma\kappa \\ det(B) &= 4\gamma^2\left(\Gamma\kappa + g^2 + \frac{\Gamma\kappa\Delta^2}{\gamma^2}\right). \end{aligned} \tag{B.16}$$

Note that the two matrices are the dressed analogues of $\mathbf{A},\mathbf{B}$ in (2.13),(2.20) of chapter 1.

By inversion of (B.6) we get the required matrix elements for calculating the force:

$$\begin{aligned} f(x) &= \langle\frac{dP}{dt}\rangle \\ &= -\hbar\nabla g Tr\left\{\rho\left(\sigma^+a + \sigma^-a^+\right)\right\} \\ &= -\hbar\nabla g\sum_{n=0}\left(\langle e,n|\sigma^+a\rho|e,n\rangle + \langle g,n+1|a^+\sigma^-\rho|g,n+1\rangle\right) \\ &\quad +\langle g,0|a^+\sigma^-\rho|g,0\rangle \\ &= -\hbar\nabla g\left(\langle e,0|\rho|g,1\rangle + c.c\right). \end{aligned}$$

Note that for the last step the sum was reduced to $n=0$ only, which is the restriction to the first manifold as was assumed above. Note also that a,a^+ act only on $|1\rangle_c,|0\rangle_c$ and thus only contribute $\sqrt{1}$ as a factor. Using eqns.(4.7) and (4.8) one finally obtains

$$\begin{aligned} f(x) &= -\hbar\frac{\nabla\omega_R}{2}\left(\frac{\omega_R}{\Omega}\rho_b - \frac{\Delta}{\Omega}\rho_g\right) \\ &= -\frac{\hbar\nabla\Omega}{2}\left(\rho_{++}-\rho_{--}\right) - \hbar\Omega\nabla\theta\left(\rho_{-+}+\rho_{+-}\right). \end{aligned} \tag{B.17}$$

Solving the matrix equations for $v = 0$ (B.6) as outlined above, one finds simple analytical expressions for populations and coherences of $|+\rangle, |-\rangle$ which are second order in η. Those may be inserted into eqn.(B.17):

$$f(x) \quad = \quad \hbar\eta^2 \frac{-\nabla g^2 \Delta_a}{|det\,(A)|^2}. \tag{B.18}$$

This is the same expression as we found in chapter 1(2.24), which is what we expected.

The same applies to the intracavity photon number, which can also be found through inversion of (B.6):

$$\begin{aligned} \langle n \rangle &= Tr\left\{\rho \mathcal{P}_{|g,1\rangle}\right\} \\ &= \langle g,1|\rho|g,1\rangle \\ &= \frac{\Omega+\Delta}{2\Omega}\rho_{++} + \frac{\Omega-\Delta}{2\Omega}\rho_{--} + \frac{\omega_R}{2\Omega}\left(\rho_{-+} + \rho_{+-}\right) \\ &= \frac{1}{2}\rho_a + \frac{\Delta}{2\Omega}\rho_b + \frac{\omega_R}{2\Omega}\rho_g. \end{aligned} \tag{B.19}$$

Again eqns.(4.7)(4.8) were used for above transformations. Inserting the solutions found for the density matrix for $v = 0$ yields for the mean photon number

$$\langle n \rangle = \eta^2 \frac{\Delta_a^2 + \Gamma^2}{|det(A)|^2}. \tag{B.20}$$

Appendix C

Higher order expectation values

In this chapter we want to come back to the question of calculating expectation values of operator products as $\langle a^+ a\sigma^- \sigma^+ \rangle$, which are needed in the calculation of the diffusion constant.

We know the following relation for the mode operators:

$$\left[a, a^+\right] = 1. \tag{C.1}$$

We assumed that the atom would only be weakly excited and thus linearised the Heisenberg equations (2.10). This linearisation is equivalent to using

$$\left[\sigma^-, \sigma^+\right] \approx \sigma^- \sigma + \approx 1. \tag{C.2}$$

Hence we have approximately the same algebra for atomic and mode operators Now we can solve the linearised equations (2.10) for steady state. That is we demand $\dot{a} = 0 = \dot{\sigma^-}$ and we obtain an expression

$$\begin{aligned} a_{st} &= C_a + \overline{F_1} \\ \sigma^-_{st} &= C_\sigma + \overline{F_2}, \end{aligned} \tag{C.3}$$

where C_a, C_σ are c numbers corresponding to the expectation values in steady state and $\overline{F_1}, \overline{F_2}$ are noise operators for which (2.11) holds as well. Instead of specifying them (which would not present a big problem) we require the commutation properties (C.1),(C.2) to be fulfilled, from which we deduce

$$\begin{aligned} \left[\overline{F_1}, \overline{F_1}^+\right] &= 1 \\ \left[\overline{F_2}, \overline{F_2}^+\right] &= 1. \end{aligned} \tag{C.4}$$

Now expectation values of combined atom-mode operator prodcuts in steady state can be calculated as follows:

$$\begin{aligned}\langle a^+a\rangle &= \langle\left(c_a^* + \overline{F_1}^+\right)\left(c_a + \overline{F_1}\right)\rangle \\ &= c_a^* c_a \\ &= \langle a^+\rangle\langle a\rangle, \end{aligned} \tag{C.5}$$

where we used property (2.11) for $\overline{F_1}, \overline{F_2}$. This can be done for all operator products where the ordering is such that a, σ^- are to the right of a^+, σ^+. So one obtains (2.21). For higher order operators products it can be done the same way,e.g.

$$\langle a^+\sigma^-\sigma^+a\rangle = \langle\left(c_a^* + \overline{F_1}^+\right)\left(c_\sigma + \overline{F_2}\right)\left(c_\sigma^* + \overline{F_2}^+\right)\left(c_a + \overline{F_1}\right). \tag{C.6}$$

Considering property (2.11) all terms containing noise operators are zero except for the term containing the noise operators of σ^-, σ^+ in antinormal order which is

$$\langle c_a^* \overline{F_2 F_2}^+ c_a\rangle. \tag{C.7}$$

Using (C.4) gives:

$$\langle c_a^* \overline{F_2 F_2}^+ c_a\rangle = c_a^* c_a = \langle a^+a\rangle, \tag{C.8}$$

where we used (C.5) for the last line. In total we get one contribution from the terms containing no noise operators at all and one contribution from one of the terms containing noise operators:

$$\langle a^+\sigma^-\sigma^+a\rangle = \langle a^+a\rangle\langle\sigma^+\sigma^-\rangle + \langle a^+a\rangle. \tag{C.9}$$

Analogously one can proceed for all operator products. This way all expectation values in steady state can be calculated very quickly from the known lower order expectation values. This is tricky but helpful and all initial conditions can be calculated in a wink. We will give an example by calculating one of the covariance initial conditions of (5.20):

$$\begin{aligned}\langle X_1(0), X_1(0)\rangle &= \langle X_1(0)X_1(0)\rangle - \langle X_1(0)\rangle\langle X_1(0)\rangle \\ &= \langle\sigma^+\sigma^-\rangle + \langle a^+a\rangle + (c_a^*c_\sigma + c_a c_\sigma^*)(c_a^*c_\sigma + c_a c_\sigma^*) \\ &\quad -\langle X_1(0)\rangle\langle X_1(0)\rangle \\ &= \langle\sigma^+\sigma^-\rangle + \langle a^+a\rangle. \end{aligned} \tag{C.10}$$

Thus one obtains for the covariances in steady state the same result as when assuming that only one photon is in the cavity and neglecting additional

terms of higher order than η^2, which was shown in (5.20). The same applies to all covariances and so the approximation that there is only one intracavity photon and additional perturbation expansion in η becomes unnecessary. **The result for the diffusion (5.23) therefore still holds for several photons inside the cavity**, provided that the saturation of the atomic transition is small. One comment is worth making: Whereas for the calculation of the force operators could in principle have been replaced by c-numbers to obtain the correct result, the noise operators caused the appearance of an additional term in the calculation of the covariances which dominates the dipole diffusion. Thus one needs a quantum treatment to obtain the correct diffusion.

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