Spontaneous Emission and Superradiance

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Abstract

An atom in an excited electronic state coupled to the electromagnetic field, will decay via spontaneous emission to a lower energy state. This thesis discusses spontaneous emission of several near atoms.

We treat the quantisation of the free electromagnetic field and dipole interaction between the field and atom, the theoretical concepts of time evolution in quantum mechanics in both, open and closed systems. We study the Schrödinger, Heisenberg and Interaction picture for closed systems, define the density matrix and develop the master equation for open systems from its time evolution.

As an example we establish the master equation in Lindblad form of an atom, where we find the spontaneous emission or Einstein A coefficient as a result.

A central result is the superradiance, the enhanced collective emission of light from several atoms, which results from the established master equation for a system of N identical atoms. The emission is discussed in the example of two atoms.

Contents

Chapter 1

Introduction

The fact that matter can spontaneously radiate light has been known for a long time in the form of Fluorescence and Luminescence. The fist theoretical description of spontaneous emission was done by Victor Weisskopf and Eugen Wigner in 1930. They used Dirac's light-theory and calculated the natural linewidth of atomic crossovers. They have shown, that an excited electronic state is not an stationary state, as we know it from Schrödinger's equation of the electron core system. The nature of this evolution is due to the coupling of the electromagnetic vacuum fluctuations to the atom [10].

By studying these light sources some other effects like superradiance, stimulated emission and absorption were discovered. At first glance superradiance and stimulated absorption look the same. But in fact they are two independent effects. The emitted radiation of a system with stimulated emission is proportional to the number of atoms and the light source's density, for superradiance only the density of atoms is relevant. This was first discussed by the theory of Dicke. This theory describes the collective spontaneous decays of more than one atom. These decays are not independent and lead to a higher photon flux for short times as the same amount of independent decays. This short time photon flux is proportional to N^2 for high numbers, where N is the number of atoms. Therefore Dicke called it an optical bomb [2].

The main reason why quantum optics was found, was a new kind of light source - the laser. Quantum optics is a field theory which discusses the interaction between electromagnetic fields and quantum mechanical systems. In quantum optics the theory of open systems has been a theme since its birth, because sources of light are open systems. In future superradiance may be used for lasers, which have a linewidth more than 1000× below the current standard. This can be used for a new generation of atomic clocks, optical lattices and laser cooling methods. But a lot of development and work is necessary to fullfill this goal.[6].

Chapter 2

Theoretical Concepts

In this chapter we present the theoretical fundament, which are needed for the understanding of the next chapters. We start with a quantum model of the free electromagnetic field. Over the time evolution of closed and open systems to the master equation of an electron interacting with vacuum fluctuations.

2.1 Quantisation of the free Electromagnetic Field

In this section the quantisation is achieved by an heuristic approach, where aspects as Lorentz covariance will not be discussed.

2.1.1 Classical Field

First we state the classical field, which is described by Maxwell's equations. If the region is free from charge ($\rho(\vec{r},t)=0$) and currents ($\vec{j}(\vec{r},t)=0$), like in the vacuum, the Maxwell equations for the electric field $\vec{E}(\vec{r},t)$ and magnetic field $\vec{B}(\vec{r},t)$ are given by

$$\mathrm{div}\vec{E}(\vec{r},t)=0,\quad \mathrm{rot}\vec{E}(\vec{r},t)=-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}, \tag{2.1a}$$

$$\mathrm{div}\vec{B}(\vec{r},t)=0,\quad \mathrm{rot}\vec{B}(\vec{r},t)=\frac{1}{c^2}\frac{\partial\vec{E}(\vec{r},t)}{\partial t}. \tag{2.1b}$$

A usual way is to define the vector potential $\vec{A}(\vec{r},t)$ and electric potential $\Phi(\vec{r},t)$. In vacuum it is very useful to use the Coulomb-gauge ($\mathrm{div}\vec{A}=0$). These potentials have to fullfill the following equations, which can be deduced from the Maxwell equations

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}(\vec{r},t)-\triangle A(\vec{r},t)=0, \tag{2.2a}$$

$$\Phi(\vec{r},t)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r},t)}{|\vec{r}-\vec{r}'|}\mathrm{d}^3r'=0. \tag{2.2b}$$

The first equation, is a wave equation, which has the general result

$$\vec{A}(\vec{r},t) = Re\left[\sum_{\vec{k},\lambda} A_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}e^{i\vec{k}\vec{r}}q_{\vec{k},\lambda}(t)\right]. \tag{2.3}$$

Here $A_{\vec{k},\lambda}$ are the amplitudes of the potential in direction of the polarisation vector $\vec{e}_{\vec{k},\lambda}$, $\vec{k}$ is the wave-vector and $q_{\vec{k},\lambda}(t)$ the time dependent term with the dimension of a length. The electric and magnetic field can be calculated from the potentials

$$\vec{E}(\vec{r},t) = -\mathrm{grad}\ \Phi - \frac{\partial\vec{A}(\vec{r},t)}{\partial t} = Re\left[\sum_{\vec{k},\lambda} A_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}e^{i\vec{k}\vec{r}}\dot{q}_{\vec{k},\lambda}(t)\right], \tag{2.4a}$$

$$\vec{B}(\vec{r},t) = \mathrm{rot}\vec{A}(\vec{r},t) = Re\left[\sum_{\vec{k}}\left(A_{\vec{k},1}|k|\vec{e}_{\vec{k},2} - A_{\vec{k},2}|k|\vec{e}_{\vec{k},1}\right)ie^{i\vec{k}\vec{r}}q_{\vec{k},\lambda}(t)\right]. \tag{2.4b}$$

For the second equation we chose two orthogonal polarisation vectors and used $\vec{e}_{\vec{k},\lambda}\cdot\vec{k} = 0$, which is a property of the e.m. field. Hence $\vec{e}_{\vec{k},1} = \vec{e}_{\vec{k},2} \times \frac{\vec{k}}{|\vec{k}|}$ and $\vec{e}_{\vec{k},2} = -\vec{e}_{\vec{k},1} \times \frac{\vec{k}}{|\vec{k}|}$. We now choose periodic boundaries appropriate for a cavity with side-length L, therefore only waves with a wave-vector $\vec{k} = (k_{x,n_x}, k_{y,n_y}, k_{z,n_z})$ with $k_{i,n_i} = n_i\frac{2\pi}{L}$, where $i = x, y, z$ and $n_i = \pm1, \pm2, \pm3, \ldots$ are allowed. The Hamiltonian H for the total field energy in the Volume L^3 is

$$\begin{aligned} H = &\int\limits_{V=L^3} \mathrm{d}V\frac{\epsilon_0}{2}\left(\left|\vec{E}(\vec{r},t)\right|^2 + c^2\left|\vec{B}(\vec{r},t)\right|^2\right) = \sum_{\vec{k},\lambda}\frac{L^3\epsilon_0}{2}\left(A^2_{\vec{k},\lambda}\dot{q}^2_{\vec{k},\lambda}(t) + c^2A^2_{\vec{k},\lambda}|\vec{k}|^2q^2_{\vec{k},\lambda}(t)\right) \\ =&\frac{1}{2}\sum_{\vec{k},\lambda} m_{\vec{k}}\dot{q}^2_{\vec{k},\lambda}(t) + m_{\vec{k}}\omega^2_{\vec{k}}q^2_{\vec{k},\lambda}(t) = \sum_{\vec{k},\lambda}\frac{p^2_{\vec{k},\lambda}(t)}{2m_{\vec{k}}} + m_{\vec{k}}\omega^2_{\vec{k}}\frac{q^2_{\vec{k},\lambda}(t)}{2}, \end{aligned} \tag{2.5}$$

with $m_{\vec{k},\lambda} = L^3\epsilon_0A^2_{\vec{k},\lambda}$ (which has the dimension of a mass), the angular velocity $\omega_{\vec{k}} = c|\vec{k}|$ and the canonical momentum $p_{\vec{k},\lambda} = \dot{q}_{\vec{k},\lambda}(t)m_{\vec{k}}$. One sees that the energy of the system is a sum of independent harmonic oscillator energies, i.e. each mode and polarisation is equivalent to a harmonic oscillator.

2.1.2 Quantisation

The quantisation can be made if we identify the $p_{\vec{k},\lambda}$ and $q_{\vec{k},\lambda}$ in equation (2.5) as operators which follow this commutator relations [8]

$$\left[q_{\vec{k},\lambda}, p_{\vec{k}',\lambda'}\right] = i\hbar\delta_{\vec{k},\vec{k}'}\delta_{\lambda,\lambda'}, \tag{2.6a}$$

$$\left[q_{\vec{k},\lambda}, q_{\vec{k}',\lambda'}\right] = 0, \qquad \left[p_{\vec{k},\lambda}, p_{\vec{k}',\lambda'}\right] = 0. \tag{2.6b}$$

It is common to make a canonical transformation to $a_{\vec{k},\lambda}$ and $a^\dagger_{\vec{k},\lambda}$

$$a_{\vec{k},\lambda} = \frac{1}{\sqrt{2m_{\vec{k}}\hbar\omega_{\vec{k}}}}\left(m_{\vec{k}}\omega_{\vec{k}}q_{\vec{k},\lambda} + ip_{\vec{k},\lambda}\right), \tag{2.7a}$$

$$a^\dagger_{\vec{k},\lambda} = \frac{1}{\sqrt{2m_{\vec{k}}\hbar\omega_{\vec{k}}}}\left(m_{\vec{k}}\omega_{\vec{k}}q_{\vec{k},\lambda} - ip_{\vec{k},\lambda}\right). \tag{2.7b}$$

In terms of $a_{\vec{k},\lambda}$ and $a^\dagger_{\vec{k},\lambda}$, equation (2.5) becomes

$$H = \hbar\sum_{\vec{k},\lambda}\omega_{\vec{k}}\left(a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda} + \frac{1}{2}\right). \tag{2.8}$$

The operators $a_{\vec{k},\lambda}$ and $a^\dagger_{\vec{k},\lambda}$ follow the commutator relations derived from (2.6a) and (2.6b)

$$\left[a_{\vec{k},\lambda}, a^\dagger_{\vec{k}',\lambda'}\right] = \delta_{\vec{k},\vec{k}'}\delta_{\lambda,\lambda'}, \tag{2.9a}$$

$$\left[a_{\vec{k},\lambda}, a_{\vec{k}',\lambda'}\right] = 0, \qquad \left[a^\dagger_{\vec{k},\lambda}, a^\dagger_{\vec{k}',\lambda'}\right] = 0. \tag{2.9b}$$

These operators are often called annihilation ($a_{\vec{k},\lambda}$) and creation ($a^\dagger_{\vec{k},\lambda}$) operator. Now, we can write the electric and magnetic field in terms of these operators

$$\vec{E}(\vec{r},t) = \sum_{\vec{k},\lambda}\mathcal{E}_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}\left(a_{\vec{k},\lambda} + a^\dagger_{\vec{k},\lambda}\right)e^{i\vec{k}\vec{r}}, \tag{2.10a}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\sum_{\vec{k},\lambda}\mathcal{E}_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}\left(a_{\vec{k},\lambda} - a^\dagger_{\vec{k},\lambda}\right)e^{i\vec{k}\vec{r}}, \tag{2.10b}$$

where $\mathcal{E}_{\vec{k},\lambda} = \left(\frac{\hbar\omega_{\vec{k}}}{2\epsilon_0 V}\right)^{\frac{1}{2}}$ has the dimenison of an electric field. The sum over the ground state energies in equation (2.8) is infinite. This is a difficulty in quantisation of the em. field. Practical experiments measuring the total energy of the field do not lead to any divergence. A detailed discussion may be found in [9]. Hence, we can choose the energy's zero point to be $E_0 = \frac{\hbar}{2}\sum_{\vec{k},\lambda}\omega_{\vec{k}}$, therefore equation (2.8) becomes

$$H = \hbar\sum_{\vec{k},\lambda}\omega_{\vec{k}}a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda}. \tag{2.11}$$

We define Fock or number states, which are needed in the following chapters. A number state satisfies the eigenvalue equation

$$a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda}\left|n_{\vec{k},\lambda}\right\rangle = n_{\vec{k},\lambda}\left|n_{\vec{k},\lambda}\right\rangle, \tag{2.12}$$

where $a^{\dagger}_{\vec{k},\lambda} a_{\vec{k},\lambda}$ is the number operator, which gives the number of photons $n_{\vec{k},\lambda}$ in the mode $(\vec{k}, \lambda)$. A number state can easily be generated from its ground state $\left|0_{\vec{k}_\lambda}\right\rangle$ (which solves $\left\langle 0_{\vec{k},\lambda}\right| a^{\dagger}_{\vec{k},\lambda} a_{\vec{k},\lambda} \left|0_{\vec{k},\lambda}\right\rangle = 0$) by using the equations $a^{\dagger}_{\vec{k},\lambda} \left|n_{\vec{k},\lambda}\right\rangle = \sqrt{n_{\vec{k},\lambda} + 1} \left|n_{\vec{k},\lambda} + 1\right\rangle$ and $a_{\vec{k},\lambda} \left|n_{\vec{k},\lambda}\right\rangle = \sqrt{n_{\vec{k},\lambda}} \left|n_{\vec{k},\lambda} - 1\right\rangle$ as

$$\left|n_{\vec{k},\lambda}\right\rangle = \frac{\left(a_{\vec{k},\lambda}\right)^{n_{\vec{k},\lambda}}}{\sqrt{(n_{\vec{k},\lambda}!}} \left|0_{\vec{k},\lambda}\right\rangle . \tag{2.13}$$

2.2 System Dynamics

In quantum mechanics there are different ways to describe the time evolution of a closed system. The expectation value of any observable does not depend on the chosen way.

2.2.1 Schrödinger Picture

In the Schrödinger picture the closed system-dynamics is given by a time-independent Hamiltonian $\hat{H}_S$. The solutions of the stationary Schrödinger equation (2.14) are called eigenstates $\left|\psi(t_0)\right\rangle_i$ to the energy (eigenvalue) E_i,

$$\hat{H}_S \left|\psi(t_0)\right\rangle_i = E_i \left|\psi(t_0)\right\rangle_i . \tag{2.14}$$

Every normalized solution is called wave-function or state vector. In the Schrödinger picture the state vector evolves with time. The time evolution of the system is given by the unitary time evolution operator $\hat{U}(t, t_0)$. This operator follows directly from the time-dependent Schrödinger-equation,

$$i\hbar \frac{\mathrm{d}}{\mathrm{dt}} \left|\psi(t)\right\rangle_i = \hat{H}_S \left|\psi(t)\right\rangle_i , \tag{2.15}$$

where $\left|\psi(t)\right\rangle_i = \hat{U}(t, t_0) \left|\psi(t_0)\right\rangle_i$ is the formal solution, with $\hat{U}(t, t_0) = \exp(-\frac{i}{\hbar}\hat{H}_S(t - t_0))$. It is common to write the time evolution operator as $\hat{U}(t)$ with the convention $t_0 = 0$. Therefore, the time evolution of an observable $\hat{A}(t)$ is $A(t) = \left\langle\phi(t)\right| \hat{A}(t) \left|\phi(t)\right\rangle$ whereas the total time derivative is given by $\frac{\mathrm{d}\hat{A}}{\mathrm{dt}} = \frac{\partial \hat{A}}{\partial t}$.
A simple example is the energy E, which is evaluated with the Hamiltonian $\hat{H}_S$. $\left\langle\eta(t)\right| \hat{H}_S \left|\eta(t)\right\rangle = \left\langle\eta(0)\right| \hat{U}(t)^{\dagger} \hat{H}_S \hat{U}(t) \left|\eta(0)\right\rangle = \left\langle\eta(0)\right| \hat{H}_S \left|\eta(0)\right\rangle = E$. We assumed that $\left|\eta(0)\right\rangle$ is an Eigenvector of $\hat{H}_S$ and used $\left[\hat{H}_S, \hat{U}(t)\right] = 0$.

2.2.2 Heisenberg Picture

The main difference between the Schrödinger picture and the Heisenberg picture is that the time evolution of an Observable is incorporated in its operator. An operator in the Heisenberg picture is defined by $\hat{A}_H(t) = \hat{U}(t)^\dagger \hat{A}(t)\hat{U}(t)$, in which $\hat{A}(t)$ is the known operator in the Schrödinger picture. The time evolution equation or Heisenberg equation is simply derived from the Schrödinger picture

$$\begin{aligned}\frac{\mathrm{d}\hat{A}_H(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{U}(t)^\dagger \hat{A}(t)\hat{U}(t)\right) = \frac{i}{\hbar}\hat{H}_S\hat{A}_H(t) - \frac{i}{\hbar}\hat{A}_H(t)\hat{H}_S + \left(\frac{\partial \hat{A}(t)}{\partial t}\right)_H \\ &= \frac{i}{\hbar}\left[\hat{H}_S, \hat{A}_H(t)\right] + \left(\frac{\partial \hat{A}(t)}{\partial t}\right)_H . \end{aligned} \tag{2.16}$$

The expectation value of an operator $\hat{A}_H(t)$ is $A(t) = \langle\phi(0)|\, \hat{A}_H(t)\, |\phi(0)\rangle =$ $= \langle\phi(0)|\, \hat{U}(t)^\dagger \hat{A}(t)\hat{U}(t)\, |\phi(0)\rangle = \langle\phi(t)|\, \hat{A}(t)\, |\phi(t)\rangle$. As you can see, the result is the same as in the Schrödinger picture. If an operator in the Schrödinger picture is time independent and commutes with the Hamiltonian, then the expectation value is a conserved quantity.

2.2.3 Interaction Picture

The Interaction or Dirac picture is very important in this thesis, because it is used to derivate the master equation and calculate the system dynamics in the following chapters. It is often used if no exact analytic solutions of the system in the Schrödinger picture is known or the Heisenberg equation can not be solved. In the Interaction picture both, state vector and observable are time-dependent [5].
First, the Hamiltonian in the Schrödinger picture is split into two parts $H = H_0 + V(t)$, where H_0 is the part where exact analytic results are known and $V(t)$ is the rest, which can depend on time. In our case it is the interaction term between the system and the reservoir (thermal bath). The idea is to make the observables time-dependent in terms of H_0, while the explicit time-dependency induced by the interaction of $V(t)$ is included in the state vectors. This can be done by defining the state vector as $|\psi(t)\rangle_D = \hat{U}^\dagger_{H_0}(t)\,|\psi\rangle$, where $\hat{U}_{H_0}(t) = \exp(-\frac{i}{\hbar}\hat{H}_0(t))$ and $|\psi\rangle = |\psi(t)\rangle$ is the state vector in the Schrödinger-picture. The transformation of an observable into the interaction picture is given by the claim that the expectation value does not change,

$$\begin{aligned}\langle\psi(t)|_D\, \hat{A}_D(t)\, |\psi(t)\rangle_D &\overset{!}{=} \langle\psi(t)|\, \hat{A}(t)\, |\psi(t)\rangle \\ &= \langle\psi(t)|_D\, \hat{U}^\dagger_{H_0}(t)\hat{A}(t)\hat{U}_{H_0}(t)\, |\psi(t)\rangle_D . \end{aligned} \tag{2.17}$$

Therefore, an operator in the interaction picture is given by $\hat{A}_D(t) = \hat{U}^\dagger_{H_0}(t)\hat{A}(t)\hat{U}_{H_0}(t)$. The dynamic of an observable can be derived similarly to the Heisenberg equation. The motion of the state vector can be calculated from its total time derivation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle_D &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}^\dagger_{H_0}(t)\right)\left|\psi(t)\right\rangle + \hat{U}^\dagger_{H_0}(t)\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle \\ &= \frac{i}{\hbar}\hat{H}_0\hat{U}^\dagger_{H_0}(t)\left|\psi(t)\right\rangle - \frac{i}{\hbar}\hat{U}^\dagger_{H_0}(t)\hat{H}\left|\psi(t)\right\rangle \\ &= \frac{i}{\hbar}\hat{U}^\dagger_{H_0}(t)\left(\hat{H}_0 - \hat{H}\right)\hat{U}_{H_0}(t)\left|\psi(t)\right\rangle_D \\ &= -\frac{i}{\hbar}\hat{V}_D(t)\left|\psi(t)\right\rangle_D\,, \qquad (2.18a)\\ \text{with } \hat{V}_D(t) &= \hat{U}^\dagger_{H_0}(t)\hat{V}(t)\hat{U}_{H_0}(t). \qquad (2.18b)\end{aligned}$$

It looks like the Schrödinger equation with the interaction term transformed into the interaction picture.

2.3 Open Quantum Systems

In reality it is impossible to isolate the system of interest from its surroundings. When solving the system of interest we always have to consider that there is an interaction between the system and its surroundings, which can affect the systems solution. Usually, we are not able to track the system and environment (or reservoir), neither theoretically nor experimentally. Hence we seek a description which gives the main influence of the surroundings, without considering the full description of the environment. This is given by the master equation, which is discussed in section 2.4. An open system consists of two parts, the system of interest S and the reservoir R. When these two systems are combined, they are a closed system and its time evolutions is given in section 2.2. The Hamiltonian of the full system is given by

$$\hat{H} = \hat{H}_S \otimes \mathbb{1}_R + V + \mathbb{1} \otimes \hat{H}_R, \qquad (2.19)$$

where the index R or S means that these is an operator in the Hilbert space of R or S. Therefore, the Hilbert space of the complete system is a product space of R and S. The operator V has parts in both spaces and gives the interaction between S and R. From now on we drop the indices and tensor products.

2.4 Master Equation

The master equation which is formally a Lindblad equation in quantum mechanics describes a non-unitary time evolution of a density operator.

2.4.1 Density Operator

The density operator is the most universal way to describe the state of a quantum mechanical system and is usually defined in the Schrödinger-picture.
The description or measurement of a state is unique, if a complete set of commuting observables (CSCO) is given or measured or given at time t. If the set of observables is incomplete, there is missing information about the system's state. This is also called a mixed state. A mixed state is a statistical ensemble with probabilities p_i, that the system is in the pure state $|\psi_i\rangle$, with $\langle\psi_i|\psi_j\rangle = \delta_{i,j}$. A pure state is reached, when a CSCO is measured at time t. Some examples for a mixed state are the spin-polarisation in the Stern-Gerlach experiment or the state of a macroscopic solid (it's not possible to measure the state of 10^{23} electrons at the same time). The density operator in a discrete basis is defined as

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| , \tag{2.20a}$$

$$\text{with } 0 \leq p_i \leq 1, \quad \sum_i p_i = 1. \tag{2.20b}$$

Hence, the expectation value of an observable $\hat{A}$ is given by

$$\begin{aligned} A &= \sum_i p_i \left\langle\psi_i\right| \hat{A} \left|\psi_i\right\rangle = \sum_i p_i \left\langle\psi_i\right| \mathbb{1}\hat{A}\mathbb{1} \left|\psi_i\right\rangle \\ &= \sum_i \sum_{m,n} p_i \left\langle\psi_i|\eta_n\right\rangle \left\langle\eta_n\right| \hat{A} \left|\eta_m\right\rangle \left\langle\eta_m|\psi_i\right\rangle \\ &= \sum_{m,n} \left\langle\eta_n\right| \hat{A} \left|\eta_m\right\rangle \sum_i p_i \left\langle\psi_i|\eta_n\right\rangle \left\langle\eta_m|\psi_i\right\rangle \\ &= \sum_{m,n} A_{m,n}\rho_{m,n} = \sum_m (A\rho)_m \\ &= Tr[\hat{A}\rho] = Tr[\rho\hat{A}] \end{aligned} \tag{2.21}$$

Tr[.] is the trace and we assume that the wave function $|\eta_i\rangle$ is an Eigenvector to $\hat{A}$. The expectation value is a statistical average over the different properties p_i, which results from the incomplete information. The time evolution of the density operator or Von Neumann equation can be derivated from the Schrödinger equation,

$$\begin{aligned} \frac{\partial\rho(t)}{\partial t} &= \sum_i p_i \left[\left(\frac{\partial}{\partial t}\left|\psi_i\right\rangle\right)\left\langle\psi_i\right| + \left|\psi_i\right\rangle \frac{\partial}{\partial t}\left\langle\psi_i\right|\right] \\ &= \sum_i p_i \left[-\frac{i}{\hbar}\hat{H}\left|\psi_i\right\rangle\left\langle\psi_i\right| + \frac{i}{\hbar}\left|\psi_i\right\rangle\left\langle\psi_i\right|\hat{H}\right] \\ &= -\frac{i}{\hbar}\left[\hat{H}, \rho(t)\right]. \end{aligned} \tag{2.22}$$

2.4.2 Master Equation

In this section, the master equation for a system coupled to a reservoir is derived. First we begin with a complete Hamiltonian $H = H_S + H_R + V$, where H_S, H_R are the Hamiltonians for the System S and Reservoir R and V is the interaction term in the Schrödinger picture. To evaluate the time evolution of the density operator $\chi(t)$ of the whole system ($S \otimes R$), we have to transform the Von Neumann equation (2.22) into the interaction picture. For this we need the unitary operator transformation used in (2.17) with $H_0 = H_S + H_R$. From now on we use the $\sim$ symbol for an operator in the interaction picture

$$\begin{aligned}
\frac{\partial}{\partial t}\chi(t) &= -\frac{i}{\hbar}\left[H(t), \chi(t)\right] \\
\frac{\partial}{\partial t}\hat{U}_{H_0}(t)\tilde{\chi}(t)\hat{U}^{\dagger}_{H_0}(t) &= -\frac{i}{\hbar}\left[\hat{U}_{H_0}(t)\tilde{H}(t)\hat{U}^{\dagger}_{H_0}(t), \hat{U}_{H_0}(t)\tilde{\chi}(t)\hat{U}^{\dagger}_{H_0}(t)\right] \\
U_{H_0}(t)\left(-\frac{i}{\hbar}\left[H_0, \tilde{\chi}(t)\right] + \frac{\partial}{\partial t}\tilde{\chi}(t)\right)U^{\dagger}_{H_0}(t) &= -\frac{i}{\hbar}\hat{U}_{H_0}(t)\left[\tilde{H}(t), \tilde{\chi}(t)\right]\hat{U}^{\dagger}_{H_0}(t) \\
-\frac{i}{\hbar}\left[H_0, \tilde{\chi}(t)\right] + \frac{\partial}{\partial t}\tilde{\chi}(t) &= -\frac{i}{\hbar}\left[H_0, \tilde{\chi}(t)\right] - \frac{i}{\hbar}\left[\tilde{V}(t), \tilde{\chi}(t)\right] \\
\frac{\partial}{\partial t}\tilde{\chi}(t) &= -\frac{i}{\hbar}\left[\tilde{V}(t), \tilde{\chi}(t)\right]. \qquad (2.23)
\end{aligned}$$

The formal integration of (2.23) gives

$$\tilde{\chi}(t) = \chi(0) + \frac{1}{i\hbar}\int_0^t dt'\left[\tilde{V}(t'), \tilde{\chi}(t')\right]. \qquad (2.24)$$

As you can see this equation can be solved by iteration. After one more iteration equation (2.24) can be written as

$$\tilde{\chi}(t) = \chi(0) + \int_0^t dt'\left(\frac{1}{i\hbar}\left[\tilde{V}(t'), \tilde{\chi}(0)\right] - \frac{1}{\hbar^2}\int_0^{t'} dt''\left[\tilde{V}(t'), \left[\tilde{V}(t''), \tilde{\chi}(t'')\right]\right]\right), \qquad (2.25)$$

and after a differentiation of the whole equation, the time evolution is

$$\dot{\tilde{\chi}}(t) = \frac{1}{i\hbar}\left[\tilde{V}(t), \tilde{\chi}(0)\right] - \frac{1}{\hbar^2}\int_0^t dt'\left[\tilde{V}(t), \left[\tilde{V}(t'), \tilde{\chi}(t')\right]\right]. \qquad (2.26)$$

Note, that this equation is exact, equation (2.22) is only cast into a form, where reasonable approximations can be made. Now, we define the reduced density operator $\rho(t)$ of the system as

$$\rho(t) = Tr_R\left[\chi(t)\right], \qquad (2.27)$$

where the trace is only taken over the reservoir. We assume that the interaction is turned on at time t=0, therefore the density operator factorizes as $\chi(0) = \rho(0)R_0$, where

R_0 is the initial reservoir density operator. Performing the partial trace in equation (2.27) we get

$$\dot{\tilde{\rho}}(t) = \frac{1}{i\hbar} Tr_R \left\{ \left[\tilde{V}(t), \tilde{\chi}(0) \right] \right\} - \frac{1}{\hbar^2} \int_0^t dt' \; Tr_R \left\{ \left[\tilde{V}(t), \left[\tilde{V}(t'), \tilde{\chi}(t') \right] \right] \right\}, \tag{2.28}$$

with $Tr_R(\tilde{\chi}) = e^{(i/\hbar)H_S t} \rho e^{(-i/\hbar)H_S t} = \tilde{\rho}$. Note, that this time evolution is non-unitary from this point on. An example is the spontaneous decay (see chapter 3) of an atom (system) in the vacuum field (reservoir). When an atomic state decays, a photon is sent out into the reservoir. Therefore the energy is not obtained. It is also not a pure state any more, because tracing over degrees of freedom means, that reservoir observables can not be evaluated. Hence, a CSCO cannot be found.

If the interaction Hamiltonian V has the specific form

$$V = \hbar \sum_i \beta_i \Gamma_i, \tag{2.29}$$

where Γ_i and β_i are proportional to lowering or rising operators of the reservoir and system, respectively, then the first term in equation (2.28) can be evaluated,

$$\begin{aligned} Tr_R \left\{ \left[\tilde{V}, \chi(0) \right] \right\} &= Tr_R \left\{ e^{(i/\hbar)H_0 t} V(t) e^{-(i/\hbar)H_0 t} \chi(0) \right\} + \dots \\ &= \sum_n \chi_n \langle u_n | V | u_n \rangle + \dots \\ &= \sum_{i,n} \langle u_n | \Gamma_i | u_n \rangle \beta_i + \dots = 0 \end{aligned} \tag{2.30}$$

where $|u_n\rangle$ is the orthonormal basis of the reservoir.

The density operator factorizes at $t = 0$ and we assume that the coupling V between S and R is weak. However, $\chi(t)$ should only show deviations in the order of V from an uncorrelated state. The reservoir is a very large system and should be therefore unaffected by the interaction. Then the density operator can be written as $\tilde{\chi}(t) = \tilde{\rho}(t) R_0 + O(V)$.

The first major approximation is a the so-called Born approximation. We neglect terms higher than second order in V, so equation (2.28) reads

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2} \int_0^t dt' \; Tr_R \left\{ \left[\tilde{V}(t), \left[\tilde{V}(t'), \tilde{\rho}(t') R_0 \right] \right] \right\}. \tag{2.31}$$

The detailed discussion of this approximation can be found in the work of Haake [3]. An important property of this equation is, that it is not Markovian, the density operator depends on its past history. The second major approximation is therefore the replacing of $\rho(t')$ by $\rho(t)$. This can be done if the correlation functions in eq. (2.34a) and (2.34b) are proportional to $\delta(t - t')$. After these two approximations (Born-Markov approximation) it looks like:

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2} \int_0^t dt' \; Tr_R \left\{ \left[\tilde{V}(t), \left[\tilde{V}(t'), \tilde{\rho}(t) R_0 \right] \right] \right\}. \tag{2.32}$$

A detailed discussion of the Markov-approximation can be found in [1] .
Therefore it becomes

$$\begin{aligned}
\dot{\tilde{\rho}}(t) &= -\sum_{i,j}\int_0^t \mathrm{d}t'\; Tr_R\left\{\left[\tilde{\beta}_i(t)\tilde{\Gamma}_i(t),\left[\tilde{\beta}_j(t')\tilde{\Gamma}_j(t'),\tilde{\rho}(t)R_0\right]\right]\right\}\\
&= -\sum_{i,j}\int_0^t \mathrm{d}t'\; Tr_R\left\{\left[\left(\tilde{\beta}_i(t)\tilde{\beta}_j(t')\tilde{\rho}(t)-\tilde{\beta}_j(t')\tilde{\rho}(t)\tilde{\beta}_i(t)\right)\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')R_0\right]\right.\\
&\quad\left.+\left[\left(\tilde{\rho}(t)\tilde{\beta}_j(t')\tilde{\beta}_i(t)-\tilde{\beta}_i(t)\tilde{\rho}(t)\tilde{\beta}_j(t')\right)\tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)R_0\right]\right\}\\
&= -\sum_{i,j}\int_0^t \mathrm{d}t'\;\left\{\left[\tilde{\beta}_i(t)\tilde{\beta}_j(t')\tilde{\rho}(t)-\tilde{\beta}_j(t')\tilde{\rho}(t)\tilde{\beta}_i(t)\right]\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\rangle_R\right.\\
&\quad\left.+\left[\tilde{\rho}(t)\tilde{\beta}_j(t')\tilde{\beta}_i(t)-\tilde{\beta}_i(t)\tilde{\rho}(t)\tilde{\beta}_j(t')\right]\langle\tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)\rangle_R\right\}, \qquad (2.33)
\end{aligned}$$

where we used the correlation functions

$$\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\rangle_R = Tr_R\left[R_0\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\right]\approx\delta(t-t'), \qquad (2.34a)$$

$$\langle\tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)\rangle_R = Tr_R\left[R_0\tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)\right]\approx\delta(t-t'). \qquad (2.34b)$$

This is the master equation which is needed for the next two chapters.
If the master equation is given in the interaction picture, the back transformation to the Schrödinger picture is

$$\dot{\rho} = \frac{1}{i\hbar}\left[H_0,\rho\right] + e^{-(i/\hbar)H_0t}\dot{\tilde{\rho}}e^{(i/\hbar)H_0t}. \qquad (2.35)$$

Chapter 3

Spontaneous Emission of a Single Atom

In the following chapter we want to derive a model for spontaneous emission of an atom, which is in a thermal bath at zero temperature. That means that the electric and magnetic fields are zero on average, but not their variances. Hence, vacuum fluctuations are interacting with the atom and causing it to decay.

3.1 Two Level Atom

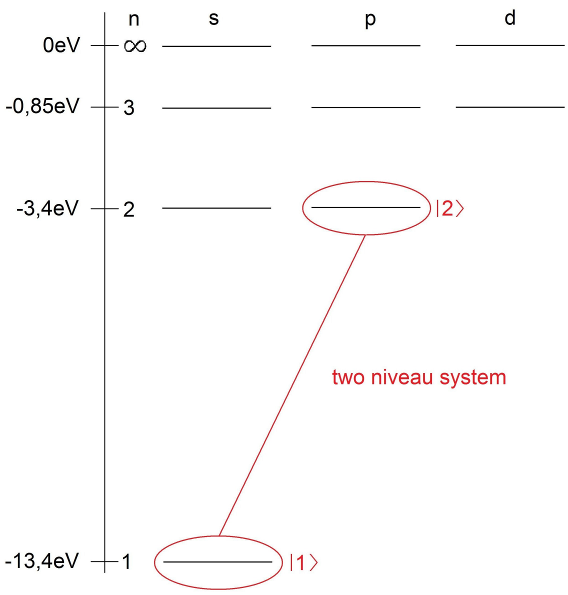

Figure 3.1: Schematic representation of the hydrogen term diagram, with a two level system

A two level system is a very important model for an atom, which is often used in quantum mechanics. It consists of the corresponding two states, the ground ($|1\rangle$) and excited ($|2\rangle$) state. These states energies E_1 and E_2, with $E_1 \leq E_2$. We consider the special case where the two levels are the two deepest bound electronic states of the valence electron in an atom (see figure 3.1).

These approximations are valid as long as the atom is prepared in state $|1\rangle$ or $|2\rangle$ (or a superposition) and no higher states can be reached. This can be achieved if the electric field is in thermal equilibrium. If it is not in thermal equilibrium, there is a chance that state $|3\rangle$ or higher can be reached, which is not part of this description.

The Hamiltonian of a two level system is given by

$$\hat{H}_A = |1\rangle \langle 1| E_1 + |2\rangle \langle 2| E_2. \tag{3.1}$$

Every two level system can mathematically be described by the Pauli spin matrices. These matrices are defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle \langle 2| + |2\rangle \langle 1| , \tag{3.2a}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\,|1\rangle \langle 2| - i\,|2\rangle \langle 1| , \tag{3.2b}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |2\rangle \langle 2| - |1\rangle \langle 1| , \tag{3.2c}$$

if we set $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore the Hamiltonian can be written as

$$\hat{H}_A = \frac{1}{2}(E_1 + E_2)\mathbb{1} + \frac{1}{2}(E_2 - E_1)\sigma_z. \tag{3.3}$$

The first term in eq. (3.3) is a constant energy shift and can eliminated if we move the atomic energies to the mean of E_1 and E_2. We identify the atomic transition frequency $\omega_A = (E_2 - E_1)/\hbar$ and write equation (3.3) as

$$\hat{H}_A = \frac{1}{2}\hbar\omega_A\sigma_z. \tag{3.4}$$

3.2 Dipole Approximation

In this section we discuss the derivation of the dipole interaction between a charged particle and the electromagnetic field.

3.2.1 Minimal Coupling

The force on a classical particle in an electromagnetic field is the Lorentz force. Hence the equation of motion is given by

$$m\ddot{\vec{r}} = q\left[\vec{E}(\vec{r}, t) + \dot{\vec{r}} \times \vec{B}(\vec{r}, t)\right]. \tag{3.5}$$

Here q is the charge of the particle and m its mass. It can be shown that the Hamilton function

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}(\vec{r}, t)\right)^2 + q\Phi(\vec{r}, t) \tag{3.6}$$

leads to (3.5). This Hamilton function can be won by replacing the kinematic momentum $\vec{p}$ by the canonical momentum $\vec{p} - q\vec{A}(\vec{r}, t)$ in the Hamilton function of a free particle and add $q\Phi(\vec{r}, t)$. This is called principle of minimal coupling.

3.2.2 Approximation

The quantisation of equation (3.6) is made by replacing $\vec{p} \to -i\hbar\vec{\bigtriangledown}$, therefore the Schrödinger equation has the form

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \left[\frac{1}{2m}\left(-i\hbar\vec{\bigtriangledown} - q\vec{A}(\hat{r},t)\right)^2 + q\Phi(\hat{r},t) + V(\hat{r})\right]\left|\psi\right\rangle . \tag{3.7}$$

We add another potential $V(\hat{r})$, which in our case is the coulomb potential of the core at the position of the electron.

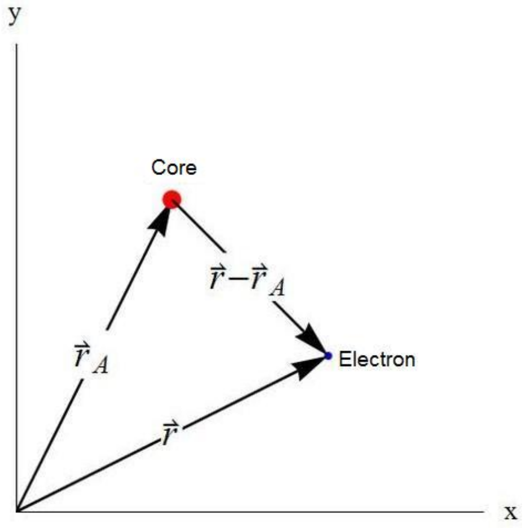

Figure 3.2: Representation of the vectors in an core-electron system.

The main approximation is to replace the position operator $\hat{r}$, which is the location of the electron by the position of the core $\vec{r}_A$, which is fixed in the terms of $\vec{A}$ and Φ (see figure 3.2). This approximation is justified if the electromagnetic field force is nearly constant over the expansion of the atom. Therefore the wavelength λ must be greater than the expansion of the atom. For example, the wavelength of visible light is about $\lambda = 500\text{nm}$ and the Bohr radius is $a_0 = 0.53\text{Å}$, hence there is a difference of about 4 orders of magnitude. Therefore, equation (3.7) becomes

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \left[\frac{1}{2m}\left(-i\hbar\vec{\bigtriangledown} - q\vec{A}(\vec{r}_A,t)\right)^2 + q\Phi(\vec{r}_A,t) + V(\hat{r})\right]\left|\psi\right\rangle . \tag{3.8}$$

The gauge freedom of classical electrodynamics can be used to

$$\vec{A}'(\vec{r},t) = A(\vec{r},t) + \text{grad}\,\Lambda(\vec{r},t), \tag{3.9a}$$

$$\Phi'(\vec{r},t) = \Phi(\vec{r},t) - \frac{\partial}{\partial t}\Lambda(\vec{r},t), \tag{3.9b}$$

where $\Lambda(\vec{r},t)$ is an arbitrary scalar function. Starting from the Coulomb gauge, then $\mathrm{div}\vec{A}(\vec{r}_A,t)=0$ and $\Phi(\vec{r}_A,t)=0$. If we choose $\Lambda(\vec{r}_A,t)=-(\vec{r}-\vec{r}_A)\vec{A}(\vec{r}_A,t)$, the vector and scalar potential becomes

$$\vec{A}'(\vec{r}_A,t)=\vec{A}(\vec{r}_A,t)-\text{grad }\vec{r}\vec{A}(\vec{r}_A,t)=0, \tag{3.10a}$$

$$\Phi'(\vec{r}_A,t)=\Phi(\vec{r}_A,t)+\frac{\partial}{\partial t}(\vec{r}-\vec{r}_A)\vec{A}(\vec{r}_A,t)=(\vec{r}-\vec{r}_A)\frac{\partial}{\partial t}\vec{A}(\vec{r}_A,t). \tag{3.10b}$$

The classical field can be calculated as $\vec{E}(\vec{r}_A,t)=-\text{grad }\Phi'(\vec{r}_A,t)-\frac{\partial}{\partial t}A'(\vec{r}_A,t)=-\text{grad }(\vec{r}-\vec{r}_A)\frac{\partial}{\partial t}\vec{A}(\vec{r}_A,t)=-\frac{\partial}{\partial t}\vec{A}(\vec{r}_A,t)$. Therefore equation (3.8) reads

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle=\left[\frac{-\hbar^2}{2m}\bigtriangleup-q(\hat{r}-\vec{r}_A)\vec{E}(\vec{r}_A,t)+V(\hat{r})\right]\left|\psi\right\rangle. \tag{3.11}$$

The second term on the right side looks like the energy of a dipole in a electric field. Hence we identify $q(\hat{r}-\vec{r}_A)$ as the dipole operator, which couples to the electric field [7].

3.3 Spontaneous Emission

Now, we want to deduce the master equation for an atom surrounded by a thermal bath at temperature T=0 (i.e. the vacuum), which acts as a reservoir of harmonic oscillators (see section 2.1.2). The first step for this has already been done in section 2.4.2.

3.3.1 Representation of an Operator

If a complete set of orthonormal states as used in (3.1), $|1\rangle$ and $|2\rangle$, is given, every Operator $\hat{A}$ can be rewritten as

$$\hat{A}=\mathbb{1}\hat{A}\mathbb{1}=\sum_{m,n}|n\rangle\langle n|\hat{A}|m\rangle\langle m|. \tag{3.12}$$

As an example we expand the dipole operator $q(\hat{r}-\vec{r}_A)=q\hat{r}'$, where $\hat{r}'$ is the distance between the electron and core in states of the two level atom

$$\begin{aligned} q\hat{r}'&=q\sum_{m,n=1}^{2}\langle n|\hat{r}'|m\rangle|n\rangle\langle m|\\ &=q\left(\langle 2|\hat{r}'|2\rangle|2\rangle\langle 2|+\langle 1|\hat{r}'|1\rangle|1\rangle\langle 1|+\langle 1|\hat{r}'|2\rangle|1\rangle\langle 2|+\langle 2|\hat{r}'|1\rangle|2\rangle\langle 1|\right)\\ &=q\left(\langle 1|\hat{r}'|2\rangle|1\rangle\langle 2|+\langle 2|\hat{r}'|1\rangle|2\rangle\langle 1|\right)=\vec{\mathrm{d}}_{1,2}\sigma_-+\vec{\mathrm{d}}_{2,1}\sigma_+. \end{aligned} \tag{3.13}$$

Here, we used $\langle 1|\hat{r}'|1\rangle=\langle 2|\hat{r}|2\rangle=0$ under the assumption that those states are symmetric, which guarantees a zero dipole moment. We also defined the dipole matrix elements, which are

$$\langle 1|\hat{r}'|2\rangle=\vec{\mathrm{d}}_{1,2}=\left(\vec{\mathrm{d}}_{2,1}\right)^*, \tag{3.14}$$

the lowering $\sigma_-=|1\rangle\langle 2|$ and rising $\sigma_+=|2\rangle\langle 1|$ operator.

3.3.2 Master Equation for a Two-Level Atom in Thermal Equilibrium

The Hamiltonian which has the form of $H = H_S + H_R + V$, with

$$H_S = \frac{1}{2}\hbar\omega_A\sigma_z, \tag{3.15a}$$

$$H_R = \sum_{\vec{k},\lambda} \hbar\omega_{\vec{k}} a^\dagger_{\vec{k},\lambda} a_{\vec{k},\lambda}, \tag{3.15b}$$

$$V = -\left(\vec{\mathrm{d}}_{1,2}\sigma_- + \vec{\mathrm{d}}^*_{1,2}\sigma_+\right)\vec{E}(\vec{r}_A, t). \tag{3.15c}$$

The atomic Hamiltonian is given from (3.4), the Hamiltonian of the reservoir is from (2.11) and the interaction term is from equation (3.11).
The interaction term has to be rewritten

$$\begin{aligned} V &= -\left(\vec{\mathrm{d}}_{1,2}\sigma_- + \vec{\mathrm{d}}^*_{1,2}\sigma_+\right)\vec{E}(\vec{r}_A, t) = \\ &= -\sum_{\vec{k},\lambda}\Big(e^{i\vec{k}r_A}\mathcal{E}_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}a_{\vec{k},\lambda}\vec{\mathrm{d}}_{2,1}\sigma_- + e^{i\vec{k}r_A}\mathcal{E}_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}a_{\vec{k},\lambda}\vec{\mathrm{d}}^*_{2,1}\sigma_+ \\ &\quad + e^{-i\vec{k}r_A}\mathcal{E}_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}a\ \dagger_{\vec{k},\lambda}\ \vec{\mathrm{d}}_{2,1}\sigma_- + e^{-i\vec{k}r_A}\mathcal{E}_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}a^\dagger_{\vec{k},\lambda}\vec{\mathrm{d}}^*_{2,1}\sigma_+\Big). \end{aligned} \tag{3.16}$$

This has to be transformed into the interaction picture of $H_S + H_R$ and changes to

$$\begin{aligned} \tilde{V} = -\sum_{\vec{k},\lambda}\Big(&e^{i\vec{k}r_A}\mathcal{E}_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}a_{\vec{k},\lambda}\vec{\mathrm{d}}_{2,1}\sigma_- e^{-i(\omega_A+\omega_{\vec{k}})} + e^{i\vec{k}r_A}\mathcal{E}_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}a_{\vec{k},\lambda}\vec{\mathrm{d}}^*_{2,1}\sigma_+ e^{-i(\omega_A-\omega_{\vec{k}})} \\ &+ e^{-i\vec{k}r_A}\mathcal{E}_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}a^\dagger_{\vec{k},\lambda}\vec{\mathrm{d}}_{2,1}\sigma_- e^{-i(-\omega_A+\omega_{\vec{k}})} + e^{-i\vec{k}r_A}\mathcal{E}_{\vec{k},\lambda}\vec{e}_{\vec{k},\lambda}a^\dagger_{\vec{k},\lambda}\vec{\mathrm{d}}^*_{2,1}\sigma_+ e^{i(\omega_A+\omega_{\vec{k}})}\Big). \end{aligned} \tag{3.17}$$

Now the rotating wave approximation has to be done, therefore all fast oscillating terms like $e^{-i(\omega_A+\omega_{\vec{k}})}$ and $e^{i(\omega_A+\omega_{\vec{k}})}$ are eliminated [4]. After the transformation back into the Schrödinger picture the interaction term becomes

$$V = \sum_{\vec{k},\lambda}\hbar\left(\kappa^*_{\vec{k},\lambda}a^\dagger_{\vec{k},\lambda}\sigma_- + \kappa_{\vec{k},\lambda}a_{\vec{k},\lambda}\sigma_+\right), \tag{3.18a}$$

$$\text{with } \kappa_{\vec{k},\lambda} = e^{i\vec{k}r_A}\sqrt{\frac{\omega_{\vec{k}}}{2\hbar\epsilon_0 V}}\left(\vec{e}_{\vec{k},\lambda}, \vec{\mathrm{d}}_{2,1}\right). \tag{3.18b}$$

We identify some terms from equation (2.29)

$$\beta_1 = \sigma_-, \qquad \beta_2 = \sigma_+ \tag{3.19a}$$

$$\Gamma_1 = \Gamma^\dagger = \sum_{\vec{k},\lambda}\kappa^*_{\vec{k},\lambda}a^\dagger_{\vec{k},\lambda}, \qquad \Gamma_2 = \Gamma = \sum_{\vec{k},\lambda}\kappa_{\vec{k},\lambda}a_{\vec{k},\lambda}. \tag{3.19b}$$

These operators must be transformed into the interaction picture:

$$\tilde{\Gamma}_1(t) = \tilde{\Gamma}^\dagger(t) = \sum_{\vec{k},\lambda} \kappa^*_{\vec{k},\lambda} a^\dagger_{\vec{k},\lambda} e^{i\omega_{\vec{k}} t}, \tag{3.20a}$$

$$\tilde{\Gamma}_2(t) = \tilde{\Gamma}(t) = \sum_{\vec{k},\lambda} \kappa_{\vec{k},\lambda} a_{\vec{k},\lambda} e^{-i\omega_{\vec{k}} t} \tag{3.20b}$$

and

$$\tilde{\beta}_1(t) = \sigma_- e^{-i\omega_A t}, \qquad \tilde{\beta}_2(t) = \sigma_+ e^{i\omega_A t}. \tag{3.21}$$

By inserting these operators into equation (2.33), where the Markov approximation is not done, we get

$$\begin{aligned} \dot{\tilde{\rho}}(t) = -\int_0^t \mathrm{d}t' \ \Big\{ &\left[\sigma_-\sigma_-\tilde{\rho}(t') - \sigma_-\tilde{\rho}(t')\sigma_-\right] e^{-i\omega_A(t+t')} \left\langle \tilde{\Gamma}^\dagger(t)\tilde{\Gamma}^\dagger(t')\right\rangle_R + \text{h.c.} \\ &+ \left[\sigma_+\sigma_+\tilde{\rho}(t') - \sigma_+\tilde{\rho}(t')\sigma_+\right] e^{i\omega_A(t+t')} \left\langle \tilde{\Gamma}(t)\tilde{\Gamma}(t')\right\rangle_R + \text{h.c.} \\ &+ \left[\sigma_-\sigma_+\tilde{\rho}(t') - \sigma_+\tilde{\rho}(t')\sigma_-\right] e^{-i\omega_A(t-t')} \left\langle \tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t')\right\rangle_R + \text{h.c.} \\ &+ \left[\sigma_+\sigma_-\tilde{\rho}(t') - \sigma_-\tilde{\rho}(t')\sigma_+\right] e^{i\omega_A(t-t')} \left\langle \tilde{\Gamma}(t)\tilde{\Gamma}^\dagger(t')\right\rangle_R + \text{h.c.} \Big\}. \end{aligned} \tag{3.22}$$

If we set the reservoir in thermal equilibrium, then the density operator $R_0 = \prod_j e^{\hbar\omega_{\vec{k}} a^\dagger a / k_\mathrm{B} T} \left(1 - e^{-\hbar\omega_{\vec{k}}/k_\mathrm{B}T}\right)$, where k_B is the Boltzmann constant. Therefore, the reservoir correlation function can explicitly be calculated

$$\left\langle \tilde{\Gamma}^\dagger(t)\tilde{\Gamma}(t')\right\rangle_R = \sum_{\vec{k},\lambda} |\kappa_{\vec{k},\lambda}|^2 e^{i\omega_{\vec{k}}(t-t')} \bar{n}(\omega_{\vec{k}}, T), \tag{3.23a}$$

$$\left\langle \tilde{\Gamma}(t)\tilde{\Gamma}^\dagger(t')\right\rangle_R = \sum_{\vec{k},\lambda} |\kappa_{\vec{k},\lambda}|^2 e^{-i\omega_{\vec{k}}(t-t')} \left[\bar{n}(\omega_{\vec{k}}, T) + 1\right], \tag{3.23b}$$

$$\left\langle \tilde{\Gamma}^\dagger(t)\tilde{\Gamma}^\dagger(t')\right\rangle_R = 0, \tag{3.23c}$$

$$\left\langle \tilde{\Gamma}(t)\tilde{\Gamma}(t')\right\rangle_R = 0, \tag{3.23d}$$

where

$$\bar{n}(\omega_{\vec{k}}, T) = \frac{e^{-\hbar\omega_{\vec{k}}/k_\mathrm{B}T}}{1 - e^{-\hbar\omega_{\vec{k}}/k_\mathrm{B}T}} \tag{3.24}$$

is the mean number of photons for an oscillator with frequency $\omega_{\vec{k}}$ at temperature T. If we consider, that we discuss this phenomenon at T=0, then only $\left\langle \tilde{\Gamma}(t)\tilde{\Gamma}^\dagger(t')\right\rangle_R$ is drifferent from zero. This can be done since $\bar{n}(\omega_{visible}, 300K) \approx 10^{-35}$, i.e. no photons are in any visible mode at ambient temperature.

The summation in these non-zero correlation functions is changed into an integral, where the number of modes in the wave vector volume of $(k{+}\mathrm{d}k)^3$ is still the same. For this we have to define the mode density

$$g(\vec{k}) = \frac{V}{8\pi^3}. \tag{3.25}$$

Making a variable change to $\tau = t - t'$ in the integral of equation (3.22), this equation becomes

$$\dot{\tilde{\rho}}(t) = -\sum_{\lambda} \int_0^t \mathrm{d}\tau \, \Big\{ \left[\sigma_+\sigma_-\tilde{\rho}(t-\tau) - \sigma_-\tilde{\rho}(t-\tau)\sigma_+\right] \\ \times \int \mathrm{d}^3k \; g(\vec{k})|\kappa(\vec{k})|^2 e^{-i(|\vec{k}|c-\omega_A)\tau} + \text{h.c.} \Big\} . \tag{3.26}$$

If we evaluate as an example the $\mathrm{d}k_x$ integral, we find an integral which is proportional to $\int_0^\infty \mathrm{d}k_x \; e^{-ik_x c\tau}$. We can expand this integral to $-\infty$, because the oscillating term will average this part to zero. Now we have a Fourier transformation of a constant function, which is proportional to $\delta(\tau)$, for a detailed discussion see [1]. Then $\tilde{\rho}(t-\tau)$ can be replaced by $\tilde{\rho}(t)$, which is called the Markov approximation. Therefore, we can deduce that

$$\dot{\tilde{\rho}}(t) = \sum_{\lambda} \left[\sigma_-\tilde{\rho}(t)\sigma_+ - \sigma_+\sigma_-\tilde{\rho}(t)\right] \int \mathrm{d}^3k \int_0^t \mathrm{d}\tau \; g(\vec{k})|\kappa(\vec{k})|^2 e^{-i(|\vec{k}|c-\omega_A)\tau} + \text{h.c.} \, . \tag{3.27}$$

We know that the last integral behaves like $\delta(\tau)$, therefore we can extend the τ integration to infinity and get

$$\lim_{t\to\infty} \int_0^t \mathrm{d}\tau \; e^{-i(|\vec{k}|c-\omega_A)\tau} = \pi\delta(|\vec{k}|c - \omega_A) + i\frac{P}{\omega_A - |\vec{k}|c}, \tag{3.28}$$

where P is the Cauchy principal value. Now we find

$$\dot{\tilde{\rho}}(t) = \left(\frac{\gamma}{2} + i\Delta\right) \left[\sigma_-\tilde{\rho}(t)\sigma_+ - \sigma_+\sigma_-\tilde{\rho}(t)\right] + \text{h.c.} \tag{3.29}$$

and

$$\gamma = 2\pi \sum_{\lambda} \int \mathrm{d}^3k \; g(\vec{k})|\kappa(\vec{k})|^2 \delta(|\vec{k}|c - \omega_A), \tag{3.30a}$$

$$\Delta = \sum_{\lambda} P \int \mathrm{d}^3k \; \frac{g(\vec{k})|\kappa(\vec{k})|^2}{\omega_A - |\vec{k}|c}. \tag{3.30b}$$

After some rearrangements we get the following equation,

$$\dot{\tilde{\rho}}(t) = -\frac{i}{2}\Delta\left[\sigma_z, \tilde{\rho}\right] + \frac{\gamma}{2}\left(2\sigma_-\tilde{\rho}(t)\sigma_+ - \sigma_+\sigma_-\tilde{\rho}(t) - \tilde{\rho}(t)\sigma_+\sigma_-\right). \tag{3.31}$$

We finally transform equation (2.35) back to the Schrödinger picture and get

$$\dot{\rho}(t) = -\frac{i}{2}\omega_A'\left[\sigma_z, \rho\right] + \frac{\gamma}{2}\left(2\sigma_-\rho(t)\sigma_+ - \sigma_+\sigma_-\rho(t) - \rho(t)\sigma_+\sigma_-\right), \tag{3.32}$$

with $\omega'_A = \omega_A + \Delta$. Note that Δ is the Lamb-Shift at temperature T=0.
This is the master equation in Lindblad form, which can be written as

$$\dot{\rho}(t) = -\frac{i}{\hbar}\left[\hat{H}'_A, \rho\right] + \mathcal{L}[\rho], \tag{3.33}$$

with $\hat{H}'_A = \frac{\hbar}{2}\omega'_A\sigma_z$ and the Lindblad superoperator $\mathcal{L}[\rho] = \frac{\gamma}{2}\left(2\sigma_-\rho(t)\sigma_+ - \sigma_+\sigma_-\rho(t) - \rho(t)\sigma_+\sigma_-\right)$.
We identify γ as the damping constant. We can solve this integration if we change to spherical coordinates in $\vec{k}$-space

$$\begin{aligned}\gamma &= 2\pi\sum_\lambda\int_0^\infty \mathrm{d}\omega\int_0^\pi \sin(\theta)\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\frac{\omega^2 V}{8\pi^2c^3}\frac{\omega}{2\hbar\epsilon_0 V}\left(\vec{e}_{\vec{k},\lambda}\cdot\vec{d}_{1,2}\right)^2\delta(\omega-\omega_A)\\ &= \frac{\omega_A^3}{8\pi^2\epsilon_0\hbar c^3}\sum_\lambda\int_0^\pi \sin(\theta)\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\left(\vec{e}_{\vec{k},\lambda}\cdot\vec{d}_{1,2}\right)^2.\end{aligned} \tag{3.34}$$

For each $\vec{k}$ we choose the orthogonal polarisation state λ_1 and λ_2, such that $\left(\vec{e}_{\vec{k},\lambda_1}\cdot\vec{d}_{1,2}\right) = 0$, then

$$\begin{aligned}\gamma &= \frac{\omega_A^3}{8\pi^2\epsilon_0\hbar c^3}\int_0^\pi \sin(\theta)\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\left(\vec{e}_{\vec{k},\lambda_2}\cdot\vec{d}_{1,2}\right)^2\\ &= d_{1,2}^2\frac{\omega_A^3}{8\pi^2\epsilon_0\hbar c^3}\int_0^\pi \sin(\theta)\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\left(1-\cos^2(\theta)\right)\\ &= \frac{1}{4\pi\epsilon_0}\frac{4\omega_A^3 d_{1,2}^2}{3\hbar c^3}.\end{aligned} \tag{3.35}$$

Now we can identify γ as the Einstein A coefficient, as obtained from the Wigner-Weisskopf theory of natural linewidth [10].
As an example we can solve the master equation from (3.32) with the denisty operator $\rho = \rho_1\left|1\right\rangle\left\langle 1\right| + \rho_2\left|2\right\rangle\left\langle 2\right|$, where $\left|2\right\rangle$ and $\left|1\right\rangle$ are the excited and ground state from section 3.1.

$$\dot{\rho}_1\left|1\right\rangle\left\langle 1\right| + \dot{\rho}_2\left|2\right\rangle\left\langle 2\right| = \frac{\gamma}{2}\left\{2\left|1\right\rangle\left\langle 1\right|\rho_2 - 2\left|2\right\rangle\left\langle 2\right|\rho_2\right\} \tag{3.36}$$

If we multiply this equation from the left and right with $\left\langle 1\right|$ and $\left|1\right\rangle$ or $\left\langle 2\right|$ and $\left|2\right\rangle$ we get two coupled differential equations

$$\dot{\rho}_2 = -\gamma\rho_2, \tag{3.37a}$$

$$\dot{\rho}_1 = \gamma\rho_2, \tag{3.37b}$$

with the solutions

$$\rho_2(t) = e^{-\gamma t}, \tag{3.38a}$$

$$\rho_1(t) = 1 - e^{-\gamma t}. \tag{3.38b}$$

Where we assumed $\rho_2(t=0) = 1$ and used the normalisation $\rho_1 + \rho_2 = 1$. The result is an exponential decay of the probabilities of states.

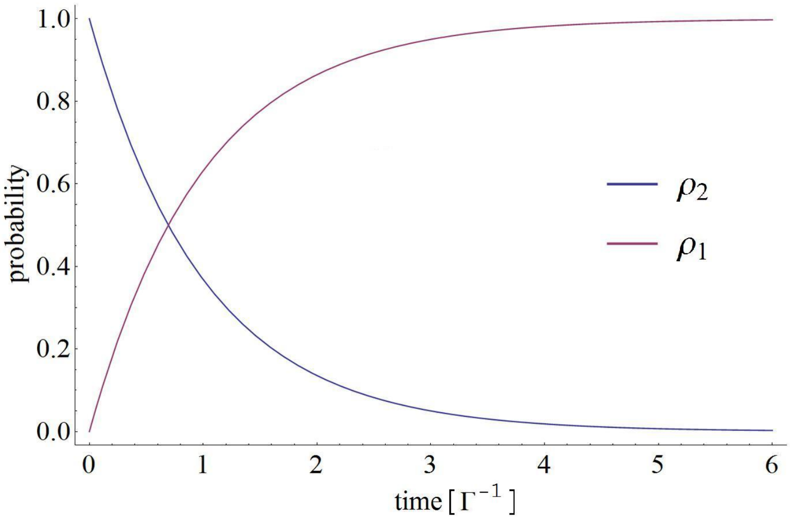

Figure 3.3: Plot of probability as a function of time, where the time is given in units of $1/\Gamma$.

Chapter 4

Collective Atom Dynamics

For want of a better term, a gas which is radiating strongly because of coherence will be called 'superradiant'.
— Robert H. Dicke, 1954 [2]

Spontaneous emission for multiple atoms can be treated independently only if their distance is large compared to the emitted wavelength. For small distances the coupling between the atoms and lightfield leads to collective effects like superradiance.

4.1 Superradiance Master Equation for N Two Level Atoms

This section contains the derivation of the master equation for superradiance effects of N identical atoms, which are surrounded by a thermal bath at temperature T=0. This derivation is quite similar to the derivation of the master equation for spontaneous emission in section 3.3.

4.1.1 Hamiltonian

The Hamiltonian for the whole system is given by $H = H_S + H_R + V$, where H_S and H_R are the system and reservoir Hamiltonian respectively, with the interaction term V. We treat these N atoms like N independent two level systems. Therefore, the system Hilbert space is a tensor product of N two level systems. The reservoir Hamilton is the same as in section 3.3 and the interaction term is the dipole coupling between each single atom and the lightfield. Hence those operators are

$$H_A = \sum_{\alpha} \frac{1}{2} \hbar \omega_A \sigma_{z,\alpha}, \tag{4.1a}$$

$$H_R = \sum_{\vec{k},\lambda} \hbar \omega_{\vec{k}} a^{\dagger}_{\vec{k},\lambda} a_{\vec{k},\lambda}, \tag{4.1b}$$

$$V = \sum_{\alpha,\vec{k},\lambda} \hbar \left(\kappa^{*}_{\alpha,\vec{k},\lambda} a^{\dagger}_{\vec{k},\lambda} \sigma_{-,\alpha} + \kappa_{\alpha,\vec{k},\lambda} a_{\vec{k},\lambda} \sigma_{+,\alpha} \right). \tag{4.1c}$$

Here, $\sigma_{z,\alpha}$, $\sigma_{-,\alpha}$ and $\sigma_{+,\alpha}$ are the Pauli spin operators for the α^{th} atom. The interaction Hamiltonian is already in the rotating wave approximation which is done in the same way as in section (3.17), with $\kappa_{\alpha,\vec{k},\lambda} = e^{i\vec{k}r_{A,\alpha}} \sqrt{\frac{\omega_{\vec{k}}}{2\hbar\epsilon_0 V}} \vec{e}_{\vec{k},\lambda} \vec{\mathrm{d}}_{2,1}$, where $\vec{r}_{A,\alpha}$ is the location of the α^{th} atom. Now we have to transform the interaction term into the interaction picture of $H_0 = H_A + H_R$ and it becomes

$$\tilde{V} = \sum_{\alpha,\vec{k},\lambda} \hbar \left(\kappa^{*}_{\alpha,\vec{k},\lambda} a^{\dagger}_{\vec{k},\lambda} \sigma_{-,\alpha} e^{-i(\omega_A - \omega_{\vec{k}})t} + \kappa_{\alpha,\vec{k},\lambda} a_{\vec{k},\lambda} \sigma_{+,\alpha} e^{i(\omega_A - \omega_{\vec{k}})t} \right). \tag{4.2}$$

4.1.2 Derivation of the Master Equation

We begin with the master equation (2.31) from Chapter 2. By inserting $\tilde{V}$ from equation (4.2), we find

$$\begin{aligned}
\dot{\tilde{\rho}} = &- \int_0^t \mathrm{d}t' \sum_{\alpha,\vec{k},\lambda} \sum_{\beta,\vec{l},\theta} \Big\{ \kappa_{\beta,\vec{l},\theta} \kappa^{*}_{\alpha,\vec{k},\lambda} \left[\sigma_{+,\beta} \sigma_{-,\alpha} \tilde{\rho}(t') - \sigma_{-,\alpha} \tilde{\rho}(t') \sigma_{+,\beta} \right] \\
&\times e^{-i(\omega_A - \omega_{\vec{k}})t'} e^{i(\omega_A - \omega_{\vec{l}})t} \left\langle a_{\vec{l},\theta} \; a^{\dagger}_{\vec{k},\lambda} \right\rangle_R \\
&+ \kappa_{\beta,\vec{l},\theta} \kappa^{*}_{\alpha,\vec{k},\lambda} \left[\tilde{\rho}(t') \sigma_{-,\alpha} \sigma_{+,\beta} - \sigma_{+,\beta} \tilde{\rho}(t') \sigma_{-,\alpha} \right] e^{-i(\omega_A - \omega_{\vec{k}})t'} e^{i(\omega_A - \omega_{\vec{l}})t} \left\langle a^{\dagger}_{\vec{k},\lambda} \; a_{\vec{l},\theta} \right\rangle_R \\
&+ \kappa^{*}_{\beta,\vec{l},\theta} \kappa_{\alpha,\vec{k},\lambda} \left[\sigma_{-,\beta} \sigma_{+,\alpha} \tilde{\rho}(t') - \sigma_{+,\alpha} \tilde{\rho}(t') \sigma_{-,\beta} \right] e^{i(\omega_A - \omega_{\vec{k}})t'} e^{-i(\omega_A - \omega_{\vec{l}})t} \left\langle a^{\dagger}_{\vec{l},\theta} \; a_{\vec{k},\lambda} \right\rangle_R \\
&+ \kappa^{*}_{\beta,\vec{l},\theta} \kappa_{\alpha,\vec{k},\lambda} \left[\tilde{\rho}(t') \sigma_{+,\alpha} \sigma_{-,\beta} - \sigma_{-,\beta} \tilde{\rho}(t') \sigma_{+,\alpha} \right] e^{i(\omega_A - \omega_{\vec{k}})t'} e^{-i(\omega_A - \omega_{\vec{l}})t} \left\langle a_{\vec{l},\theta} \; a^{\dagger}_{\vec{k},\lambda} \right\rangle_R \Big\},
\end{aligned} \tag{4.3}$$

which we already used in (3.23c) and (3.23d), such that the correlation functions $\left\langle a^{\dagger}_{\vec{l},\theta} \; a^{\dagger}_{\vec{k},\lambda} \right\rangle_R$ $= \left\langle a_{\vec{l},\theta} \; a_{\vec{k},\lambda} \right\rangle_R = 0$. In equation (2.30) we have shown that the correlation functions $\left\langle a_{\vec{l},\theta} \; a^{\dagger}_{\vec{k},\lambda} \right\rangle_R = \left\langle a^{\dagger}_{\vec{l},\theta} \; a_{\vec{k},\lambda} \right\rangle_R = 0$ for $\vec{k} \neq \vec{l}$ and $\lambda \neq \theta$. The remaining correlation functions can be evaluated as

$$\left\langle a_{\vec{k},\lambda} \; a^{\dagger}_{\vec{k},\lambda} \right\rangle_R = \bar{n}(\omega_{\vec{k}}, T) + 1, \tag{4.4a}$$

$$\left\langle a^{\dagger}_{\vec{k},\lambda} \; a_{\vec{k},\lambda} \right\rangle_R = \bar{n}(\omega_{\vec{k}}, T). \tag{4.4b}$$

Here $\bar{n}(\omega_{\vec{k}}, T)$ is the same as used in (3.24) and is the mean photon number in the k^{th} mode. Now we consider, that the temperature is $T = 0$, then $\bar{n}(\omega_{\vec{k}}, T) = 0$ and only $\langle a_{\vec{k},\lambda}\, a^\dagger_{\vec{k},\lambda}\rangle_R$ is not zero. Therefore $\dot{\tilde{\rho}}$ becomes

$$\begin{aligned}\dot{\tilde{\rho}} = &-\int_0^t \mathrm{d}t' \sum_{\alpha,\beta,\vec{k},\lambda} \kappa_{\beta,\vec{k},\lambda}\kappa^*_{\alpha,\vec{k},\lambda} \Big\{\left[\sigma_{+,\beta}\sigma_{-,\alpha}\tilde{\rho}(t') - \sigma_{-,\alpha}\tilde{\rho}(t')\sigma_{+,\beta}\right] e^{i(\omega_A-\omega_{\vec{k}})(t-t')} \\ &+ \left[\tilde{\rho}(t')\sigma_{+,\alpha}\sigma_{-,\beta} - \sigma_{-,\beta}\tilde{\rho}(t')\sigma_{+,\alpha}\right] e^{-i(\omega_A-\omega_{\vec{k}})(t-t')}\Big\}. \qquad (4.5)\end{aligned}$$

Now, the summation over $\vec{k}$ is changed into an integral, with the mode density $g(\vec{k}) = \frac{V}{8\pi^3}$ from equation (3.25)

$$\begin{aligned}\dot{\tilde{\rho}} = &-\int_0^t \mathrm{d}t' \int \mathrm{d}^3k\; g(\vec{k}) \sum_{\alpha,\beta,\lambda} \kappa_{\beta,\vec{k},\lambda}\kappa^*_{\alpha,\vec{k},\lambda} \Big\{\left[\sigma_{+,\beta}\sigma_{-,\alpha}\tilde{\rho}(t') - \sigma_{-,\alpha}\tilde{\rho}(t')\sigma_{+,\beta}\right] e^{i(\omega_A-|\vec{k}|c)(t-t')} \\ &+ \left[\tilde{\rho}(t')\sigma_{+,\alpha}\sigma_{-,\beta} - \sigma_{-,\beta}\tilde{\rho}(t')\sigma_{+,\alpha}\right] e^{-i(\omega_A-|\vec{k}|c)(t-t')}\Big\}. \\ = &-\int_0^t \mathrm{d}\tau \int_0^\infty \mathrm{d}\omega\; \frac{\omega^2 V}{8\pi^3 c^3} \sum_{\alpha,\beta,\lambda} \int \mathrm{d}\Omega\; \kappa_{\beta,\vec{k},\lambda}\kappa^*_{\alpha,\vec{k},\lambda} \\ &\times \Big\{\left[\sigma_{+,\beta}\sigma_{-,\alpha}\tilde{\rho}(t-\tau) - \sigma_{-,\alpha}\tilde{\rho}(t-\tau)\sigma_{+,\beta}\right] e^{i(\omega_A-\omega)\tau} \\ &+ \left[\tilde{\rho}(t-\tau)\sigma_{+,\alpha}\sigma_{-,\beta} - \sigma_{-,\beta}\tilde{\rho}(t-\tau)\sigma_{+,\alpha}\right] e^{-i(\omega_A-\omega)\tau}\Big\}. \qquad (4.6)\end{aligned}$$

In the last step we choose spherical coordinates for the integral over $\vec{k}$ space and a variable change to $\tau = t - t'$. We have shown in section 3.3, that the correlation functions are proportional to $\delta(\tau)$. Hence, the Markov approximation can be made by replacing $\tilde{\rho}(t-\tau)$ with $\rho(t)$. Now (4.6) can be restated as

$$\begin{aligned}\dot{\tilde{\rho}} = &-\int_0^\infty \mathrm{d}\omega\; \frac{\omega^2 V}{8\pi^3 c^3} \sum_{\alpha,\beta,\lambda} \int \mathrm{d}\Omega\; \kappa_{\beta,\vec{k},\lambda}\kappa^*_{\alpha,\vec{k},\lambda} \\ &\times \Big\{\left[\sigma_{+,\beta}\sigma_{-,\alpha}\tilde{\rho}(t) - \sigma_{-,\alpha}\tilde{\rho}(t)\sigma_{+,\beta}\right] \int_0^t \mathrm{d}\tau\; e^{i(\omega_A-\omega)\tau} \\ &+ \left[\tilde{\rho}(t)\sigma_{+,\alpha}\sigma_{-,\beta} - \sigma_{-,\beta}\tilde{\rho}(t)\sigma_{+,\alpha}\right] \int_0^t \mathrm{d}\tau\; e^{-i(\omega_A-\omega)\tau}\Big\}. \qquad (4.7)\end{aligned}$$

The spherical surface integral can be evaluated by choosing the polarisation vector $\vec{e}_{\lambda,1}$ orthogonal to $\vec{d}_{2,1}$:

$$\begin{aligned}&\frac{V}{8\pi^3 c^3}\sum_\lambda \int \mathrm{d}\Omega\; \kappa_{\beta,\vec{k},\lambda}\kappa^*_{\alpha,\vec{k},\lambda} = \frac{1}{4\pi\epsilon_0}\frac{\omega}{\pi c^3\hbar}\int \frac{\mathrm{d}\Omega}{4\pi}\left(|\vec{d}_{2,1}|^2 - \left(\vec{e}_{\lambda,2}\cdot\vec{d}_{2,1}\right)^2\right) e^{i\vec{k}(\vec{r}_{A,\beta}-\vec{r}_{A,\alpha})} \\ &= \frac{1}{4\pi\epsilon_0}\frac{\omega}{\pi c^3\hbar}\left(\frac{\sin\xi}{\xi}\left(1-\left(\hat{\vec{d}}_{2,1}\cdot\hat{\vec{r}}_{\beta,\alpha}\right)^2\right) + \left(1-3\left(\hat{\vec{d}}_{2,1}\cdot\hat{\vec{r}}_{\beta,\alpha}\right)^2\right)\left(\frac{\cos\xi}{\xi^2}-\frac{\sin\xi}{\xi^3}\right)\right) \\ &= \frac{1}{4\pi\epsilon_0}\frac{\omega}{\pi c^3\hbar}F(\xi), \qquad (4.8)\end{aligned}$$

with $\xi = |\vec{k}||\vec{r}_{\beta,\alpha}|$, $\vec{r}_{A,\beta} - \vec{r}_{A,\alpha}$, $\hat{\vec{d}}_{2,1} = \frac{\vec{d}_{2,1}}{|\vec{d}_{2,1}|}$ and $\hat{\vec{r}}_{\beta,\alpha} = \frac{\vec{r}_{\beta,\alpha}}{|\vec{r}_{\beta,\alpha}|}$. The time integral was already evaluated in (3.28) . By using the spontaneous emission constant γ from (3.35), equation (4.7) becomes

$$\begin{aligned}\dot{\tilde{\rho}} =& \int_0^\infty \mathrm{d}\omega \sum_{\alpha,\beta} \frac{3\omega^3\gamma}{4\omega_A^3\pi} F\left(\frac{\omega}{c}\ |\vec{r}_{\beta,\alpha}|\right) \\ &\times \left\{ \left[2\sigma_{-,\alpha}\tilde{\rho}(t)\sigma_{+,\beta} - \sigma_{+,\beta}\sigma_{-,\alpha}\tilde{\rho}(t) - \tilde{\rho}(t)\sigma_{+,\beta}\sigma_{-,\alpha}\right]\pi\delta(\omega-\omega_A) \right. \\ &+ \left. \left[\tilde{\rho}(t)\sigma_{+,\alpha}\sigma_{-,\beta} - \sigma_{+,\alpha}\sigma_{-,\beta}\tilde{\rho}(t)\right] i \frac{P}{\omega_A - \omega} \right\} \end{aligned} \tag{4.9}$$

For the second term we use

$$\Delta^{\pm}_{\alpha,\beta} = \int_0^\infty \mathrm{d}\omega\ \frac{3\omega^3\gamma}{4\omega_A^3\pi} F\left(\frac{\omega}{c}\ |\vec{r}_{\beta,\alpha}|\right) \frac{P}{\omega_A \pm \omega}. \tag{4.10}$$

The integral of the first term can be evaluated and equation (4.9) changes to

$$\begin{aligned}\dot{\tilde{\rho}} =& \sum_{\alpha,\beta} \left\{ \frac{3\gamma}{4} F\left(\frac{\omega_A}{c}\ |\vec{r}_{\beta,\alpha}|\right) \left[2\sigma_{-,\alpha}\tilde{\rho}(t)\sigma_{+,\beta} - \sigma_{+,\beta}\sigma_{-,\alpha}\tilde{\rho}(t) - \tilde{\rho}(t)\sigma_{+,\beta}\sigma_{-,\alpha}\right] \right. \\ &+ \left. i\left(\Delta^-_{\alpha,\beta} + \Delta^+_{\alpha,\beta}\right)\left[\tilde{\rho}(t), \sigma_{+,\alpha}\sigma_{-,\beta}\right] \right\}. \end{aligned} \tag{4.11}$$

Note, that the rotating wave approximation does not give the correct result for the Lamb shift, therefore $\Delta^+_{\alpha,\beta}$ must be included [1]. After some conversions, (4.11) is given by

$$\begin{aligned}\dot{\tilde{\rho}} =& \sum_{\alpha,\beta} \frac{\Gamma_{\alpha,\beta}}{2} \left[2\sigma_{-,\alpha}\tilde{\rho}(t)\sigma_{+,\beta} - \sigma_{+,\beta}\sigma_{-,\alpha}\tilde{\rho}(t) - \tilde{\rho}\sigma_{+,\beta}\sigma_{-,\alpha}\right] \\ &+ i \sum_{\alpha,\beta,\alpha\neq\beta} \frac{\Delta_{\alpha,\beta}}{2} \left[\tilde{\rho}(t), \sigma_{+,\alpha}\sigma_{-,\beta}\right] + i\sum_{\alpha} \Delta_\alpha \left[\rho(t), \sigma_{z,\alpha}\right]. \end{aligned} \tag{4.12}$$

We defined $\Gamma_{\alpha,\beta} = \frac{3\gamma}{2} F\left(\frac{\omega}{c}|\vec{r}_{\beta,\alpha}|\right)$, where $\Gamma_{\alpha,\alpha} = \gamma$ and $\Delta_{\alpha,\beta} = \Delta^+_{\alpha,\beta} + \Delta^-_{\alpha,\beta}$. Now, we have to transform the master equation back into the Schrödinger picture with equation (2.35) and get

$$\dot{\rho}(t) = -\frac{i}{\hbar}\left[H'_S, \rho(t)\right] + \mathcal{L}[\rho], \tag{4.13}$$

with

$$H'_S = \sum_\alpha \frac{1}{2}\hbar\left(\omega_A + \Delta_{\alpha,\alpha}\right)\sigma_{z,\alpha} + \sum_{\alpha,\beta,\alpha\neq\beta} \hbar\Delta_{\alpha,\beta}\sigma_{+,\beta}\sigma_{-,\alpha} \tag{4.14}$$

and the Lindblad Superoperator

$$\mathcal{L}[\rho] = \sum_{\alpha,\beta} \frac{\Gamma_{\alpha,\beta}}{2} \left[2\sigma_{-,\alpha}\rho(t)\sigma_{+,\beta} - \sigma_{+,\beta}\sigma_{-,\alpha}\rho(t) - \rho(t)\sigma_{+,\beta}\sigma_{-,\alpha}\right]. \tag{4.15}$$

As an example we plot $\Gamma_{\alpha,\beta}$, for $\alpha \neq \beta$ and $\vec{r}_{\alpha,\beta} \cdot \vec{d}_{1,2} = 0$ as an function of $|\vec{r}_{\beta,\alpha}|$ in figure 4.1.

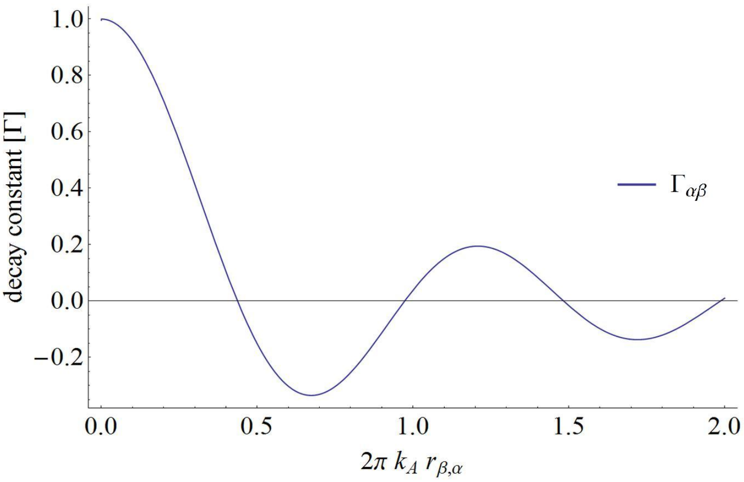

Figure 4.1: $\Gamma_{\alpha,\beta}$, for $\alpha \neq \beta$ as an function of $|\vec{r}_{\beta,\alpha}|$, with $\vec{r}_{\alpha,\beta} \cdot \vec{d}_{1,2} = 0$.

4.2 Example for Two Two-Level Atoms

In this section we solve the master equation from (4.13) for two identical atoms and discuss its photon flux.

In the product Hilbert space of two atoms a complete orthonormal basis is given by $|22\rangle = |2\rangle_1 \otimes |2\rangle_2\,, |21\rangle = |2\rangle_1 \otimes |1\rangle_2\,, |12\rangle = \dots\,, |11\rangle = \dots$, where the indices denote the first or second atom and $|1\rangle\,,\ |2\rangle$ are the state vectors of the two level system from section 3.1. It can be shown, that these vectors are no eigenvectors to H'_S from (4.14). Therefore we define one triplet

$$|1,1\rangle = |22\rangle\,, \tag{4.16a}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|21\rangle + |12\rangle\right), \tag{4.16b}$$

$$|1,-1\rangle = |11\rangle \tag{4.16c}$$

and a singlet state as

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|21\rangle - |12\rangle\right), \tag{4.16d}$$

which are eigenvectors of H'_S. A schematic representation, of the eigenvalues can be seen in figure 4.2.

Now, we define the projecting operator as

$$P_{rm,r'm'} = |rm\rangle\,\langle r'm'|\,, \tag{4.17}$$

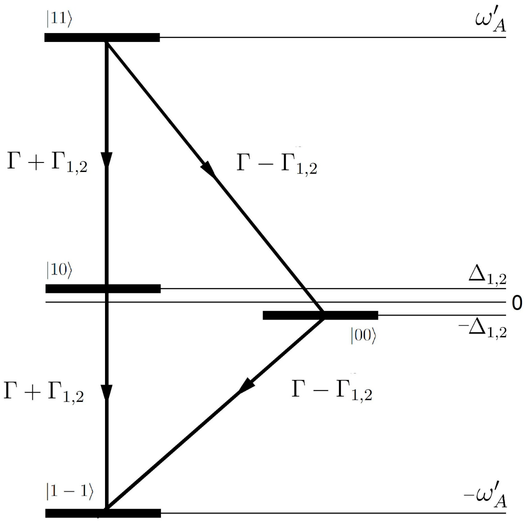

Figure 4.2: Schematic representation of the state's energies and their decay rates.

which satisfy

$$\langle P_{rm,r'm'}\rangle = \rho_{rm,r'm'}(t). \tag{4.18}$$

These operators fulfil the following identities,

$$\sigma_{+,1}\sigma_{-,1} + \sigma_{+,2}\sigma_{-,2} = P_{10,10} + P_{00,00} + 2P_{11,11}, \tag{4.19a}$$

$$\sigma_{+,1}\sigma_{-,2} = \frac{1}{2}\left(P_{10,10} - P_{00,00} - P_{00,10} + P_{10,00}\right). \tag{4.19b}$$

Hence, we can rewrite equation (4.13) where we use (4.18) and therefore get four coupled differential equations for the diagonal elements of the reduced density operator

$$\dot{\rho}_{11} = -2\Gamma\rho_{11}, \tag{4.20a}$$

$$\dot{\rho}_{10} = \left(\Gamma+\Gamma_{1,2}\right)\rho_{11} - \left(\Gamma+\Gamma_{1,2}\right)\rho_{10}, \tag{4.20b}$$

$$\dot{\rho}_{00} = \left(\Gamma-\Gamma_{1,2}\right)\rho_{11} - \left(\Gamma-\Gamma_{1,2}\right)\rho_{00}, \tag{4.20c}$$

$$\dot{\rho}_{1-1} = \left(\Gamma+\Gamma_{1,2}\right)\rho_{10} + \left(\Gamma-\Gamma_{1,2}\right)\rho_{00}. \tag{4.20d}$$

The emission rates are shown in figure 4.2 and the numeric solution for the coefficients in figure 4.3. For this example we assumed $\left(\vec{r}_{1,2}\cdot\vec{d}_{1,2}\right) = 0$,$\rho_{22}(t=0) = 1$, which can be achieved by a short π-pulse and $|\vec{r}_{1,2}|\frac{\omega_A}{c} = 0.6\ \pi$.

If $\Gamma+\Gamma_{1,2} > \Gamma$ or $\Gamma-\Gamma_{1,2} > \Gamma$, then this decay channel it is called superradiant, if its below Γ it is subradiant, see figure 4.1, where $\alpha = 1$ and $\beta = 2$. In the case, that $\Gamma_{1,2}$ is nearly Γ, then the state $|00\rangle$ is extreme stable and is therefore subradiant.

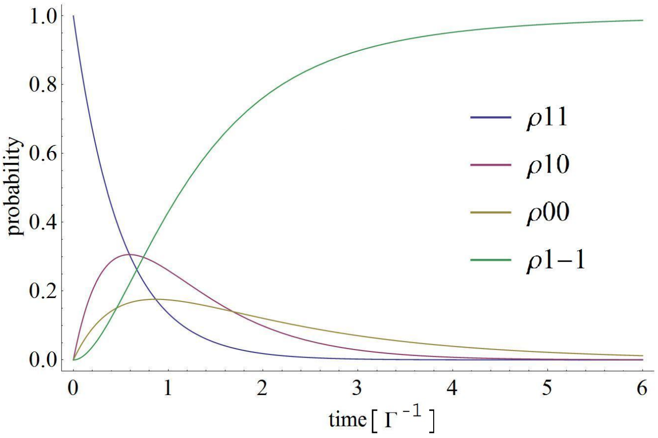

Figure 4.3: Numerical plot of probabilities for superradiance effects as a function of time, where the time is given in units of $1/\Gamma$.

Now, we can calculate the number of emitted photons. We know, that every time an electron decays, a photon is emitted. Hence, the photon flux I is proportional to $\frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha \langle \sigma_{+,\alpha}\sigma_{-,\alpha}\rangle$. Therefore we get $I \propto 2\Gamma\rho_2$ for the photon flux for spontaneous emission of two independent atoms (see (3.38a)). For the supperradiant case we get $I \propto 2\Gamma\rho_{22} + (\Gamma + \Gamma_{1,2})\rho_{10} + (\Gamma - \Gamma_{1,2})\rho_{10}$. As an example, where $|\vec{r}_{1,2}|\omega_A/c = 2\pi\ 10^{-1}$, we get a higher photon flux due to superradiant than with spontaneous emission of independent atoms (see figure 4.4).

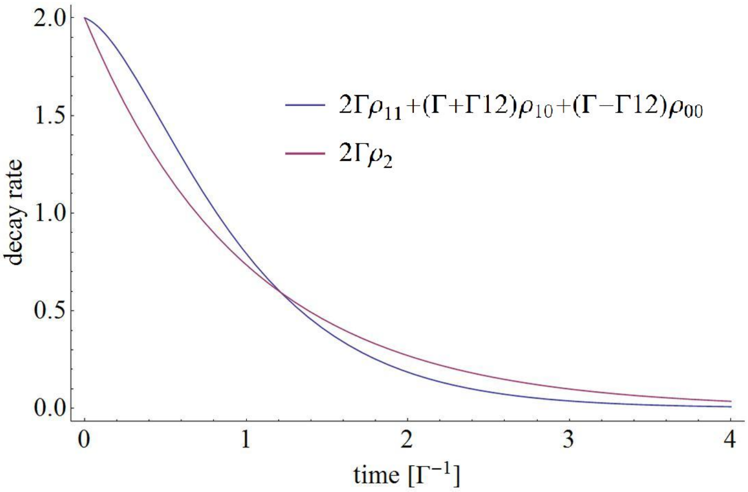

Figure 4.4: Decay rate (which is proportional to the photon flux) for superradiance and spontaneous emission as a function of time, where the time is given in units of $1/\Gamma$.

Chapter 5

Conclusions

As a result of the quantisation of the free electromagnetic field, it was shown, that each classical mode is equivalent to an harmonic oscillator. With the result of the dipole approximation it was possible to describe the interaction of the atoms and field.

We specially discussed the time evolution of the density operator of open systems, where we made a Born Markov approximation. This master equation was solved for the spontaneous emission of one atom. As a result we got the exponential decay of the excited state.

Further we derived the master equation for an ensemble of N identical atoms and solved it for the case of two atoms.From the discussion of the decay rates,we learned that the superradiant system's photon flux is higher for short times,then the system with spontaneous decay.

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