

A Superradiant Laser

Bachelorarbeit

zur Erlangung des akademischen Grades
Bachelor of Science
(BSc)

eingereicht an der Fakultät für
Mathematik, Informatik und Physik
der Universität Innsbruck

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Innsbruck, am 19.Juli.2019

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1 Introduction

Since their invention lasers have become an important tool in physics. For emitting a strong, directional beam of laser light, the process of the so-called stimulated emission of photons is used: When an excited atom decays and emits a photon, this photons can stimulate other atoms to emit additional photons, hence the acronym LASER (Light Amplification by Stimulated Emission of Radiation).

One of the most important properties of a laser is its small linewidth. With the almost monochromatic laser light a lot of physical applications are possible: detecting gravity waves, defining the second, groundstate cooling and quantum manipulation of nanomechanical objects, and quantum computation [1], or the construction of atomic clocks to measure time with great precision. From there, there are a lot of different applications, which rely on measuring time with great precision: GPS, synchronization of data and communication networks, precise measurements of the gravitational potential of the earth, radio astronomy, tests of theories of gravity, and tests of the fundamental laws of physics [11]. However, the main problem in increasing the stability of the atomic clocks are the lasers which interrogate the ultranarrow clock-transitions. The linewidth of these lasers is mainly limited by the thermal noise of the cavity mirrors [11]. By using a laser operating in the “bad-cavity”-regime, also called superradiant laser, one could significantly reduce the influence of the thermal noise on the linewidth [1]. These superradiant lasers have bad mirrors, which leads to very small average photon numbers in the cavity and therefore only weak influence of the thermal noise of the cavity mirrors. The phase information, which is carried by the photons in a standard laser, is now stored mainly in the atoms [1].

The significantly decreased linewidth of these lasers allow for a variety of different applications. For example, they could be used for cooling and trapping of dipole interacting clock-atoms [8]. It has even been suggested, that superradiant lasers may lead to searches for physics beyond the standard model [1].

2 Outline

To understand the basic concept of a laser, one has to consider three important processes: Absorption, spontaneous and stimulated emission of photons by atoms. The atom has in general several discrete energy levels and corresponding energy differences. If a photon with an energy close to such an energy difference hits the atom, it can get absorbed and the atom changes its state from the ground state to an excited state. Similar to radioactive decay, these excited atoms decay with a certain lifetime into the ground state by emitting such a photon. This is called spontaneous emission, and it ensures energy conservation as the energy of the photon is the energy of the transition. The concept of a laser is based on a third process: the stimulated emission. When atoms are in an excited state, an incoming photon with matching energy can stimulate the transition into the ground state and therefore the stimulated emission of another photon. The two photons now have the same energy and the same phase. Each of these two photons can stimulate the emission of another photon by changing the state of an atom from the excited state to the ground state. By this process of stimulated emission there can emerge a cascade of photons, all with the same frequency and phase information. However, one must make sure that the number of atoms in the excited state exceeds the number of atoms in the ground state (population inversion), as otherwise the probability for absorption is higher than the probability of stimulated emission and the cascade can not emerge.

To keep it simple, we want to restrict ourselves to a two-level-atom. However, we can not achieve population inversion using light in a mere two-level-system. If we coherently pumped the atoms, we would get Rabi-oscillations, where we would maximally achieve an equal occupation of ground and excited state averaged over time. However, if one takes the spontaneous emission into account (which is not considered when deriving the Rabi-oscillations), we would not even achieve equal occupation. Moreover, we can not achieve population inversion by putting the atoms in a thermal reservoir, as in thermal equilibrium higher energy states have lower probabilities due to the Boltzmann law. This can also be understood by looking at the einstein rate equations [4]:

$$\frac{dN_2}{dt} = N_1 B_{12} \rho(\omega_{12}) - N_2 B_{21} \rho(\omega_{12}) - N_2 \gamma \quad (2.1)$$

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt} \quad (2.2)$$

Here, N_1 denotes number of atoms in the ground state and N_2 denotes the number of atoms in the excited state. $\rho(\omega_{12})$ is the energy density at the frequency ω_{12} , where ω_{12} is the frequency corresponding to the energy difference of the transition., the time is

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denoted by t . B_{12} and B_{21} are the einstein B-coefficients, which describe the rate per energy density ρ of a transition from the ground state to the excited state (B_{12}) and the reversed process (B_{21}). Assuming that both levels are equally degenerate, we can simplify $B_{12} = B_{21} := B$ [4]. The spontaneous emission from the excited state into the ground state is given by γ . As there is no spontaneous emission from the ground state into the excited state, there is no need for a term $+N_1\gamma$. The change of the number of the atoms in the excited state $\frac{dN_2}{dt}$ is therefore given by the rate at which atoms get excited by the thermal equilibrium ($N_1B\rho(\omega_{12})$) minus the rate at which the atoms get relaxed ($N_2B\rho(\omega_{12})$) minus the rate of spontaneous decay ($N_2\gamma$). The second equation follows simply from demanding that $N_1 + N_2$ is a conserved quantity. We calculate the steady state solution by demanding $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$, which trivially solves equation 2.2. From equation 2.1 we get

$$\frac{N_2}{N_1} = \frac{B\rho(\omega_{12})}{B\rho(\omega_{12}) + \gamma} \quad (2.3)$$

For population inversion to occur, we would need $\frac{N_2}{N_1} > 1$. But looking at equation 2.3 one can see, that this can never be achieved, even with very high energy density $\rho(\omega_{12})$. So how can we achieve population inversion in a two-level-system? The answer is by introducing a third level above the ground state $|g\rangle$ and excited state $|e\rangle$, the auxiliary state $|a\rangle$. This is depicted in figure 2.1 on the left side. The idea is the following: If we transfer the atoms from the ground state $|g\rangle$ to the auxiliary state $|a\rangle$ with a rate R^0 , then the atoms in the auxiliary state very rapidly decay into the excited state $|e\rangle$ with a rate of α . If the rate α is large compared to R^0 and γ , then there will be effectively no atoms in the auxiliary state $|a\rangle$, as these very rapidly decay into the excited state $|e\rangle$. Moreover, the excitation of the atoms from the ground state over the auxiliary state to the excited state can be understood as a spontaneous “decay” from the ground state into the excited state with a rate R . In conclusion we can simplify the three-level-system to an effective two-level-system with spontaneous “decay” from the ground state into the excited state with a rate R , this is depicted in the right side of figure 2.1. As we can choose R^0 and this influences R , we can modify the rate R in the experiment. In the following, we will denote R as the pump rate, as it is the rate at which we “pump” the atoms from the ground state into the excited state. In the einstein rate equations we take the pump rate R into account by adding an additional term N_1R to the right-hand-side of equation 2.1. This results in an additional R in the numerator of the right-hand-side of equation 2.3. Choosing R to be greater than γ this eventually results in population inversion.

To model the laser, we also need to consider a cavity, in which the photons can travel. This is depicted in figure 2.2. By achieving population inversion, there will be a cascade of photons being emitted from the atoms. At the end of the cavity there are mirrors, which reflect the photons, so that the photons can stimulate emission again. This leads to a certain light field in the cavity. By having one mirror not completely reflecting all photons, some of the photons can leak from the cavity. These leaked photons make up the laser light coming from the cavity. In figure 2.2 there is one two-level-system with

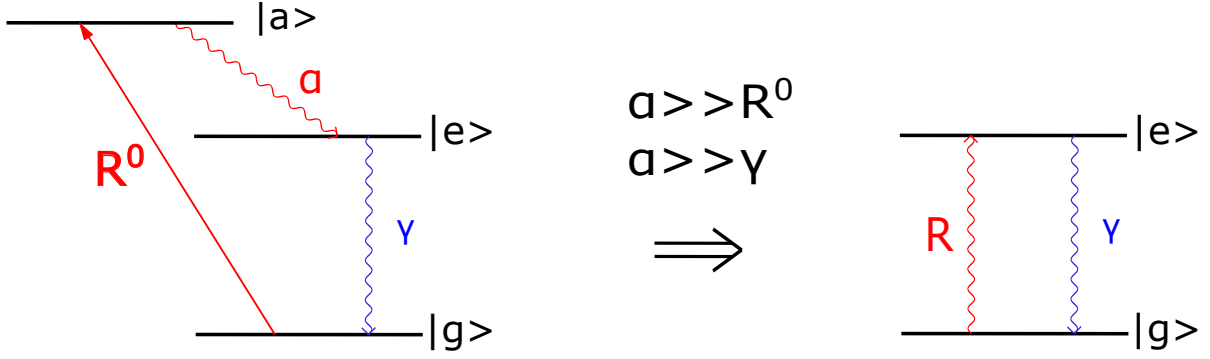


Figure 2.1: Achieving population inversion with a two-level-system. When α is much greater than R^0 and γ , this results in almost no occupation of the auxiliary state. The atoms change from the ground state into the auxiliary state, and from there into the excited state, this can be understood as a spontaneous “decay” from the ground state into the excited state. In conclusion, the three-level-system can be understood as an effective two-level-system with a pump rate R from the ground state into the excited state.

ground state $|g\rangle$ and excited state $|e\rangle$ shown. The two-level-system interacts with the light field, which is depicted by the black arrows between the two levels, this happens with a rate g . To describe this process, we will use the Jaynes-Cummings-Model, which will be derived in section 3.1. Moreover, the two-level-system is influenced by the spontaneous decay from the excited state to the ground state with rate γ and the pump rate R , which transfers the atoms from the ground state into the excited state. These processes can be accounted for by using the master-equation approach. On the left side there is the mirror with 100% reflectivity (the bold line), on the right side the dashed line indicates a mirror, whose reflectivity is less than 100%. Therefore photons leak from the cavity with a certain rate κ , this process can be also described by using a master-equation approach.

The goal of the following chapter 3 will be to describe the setup depicted in figure 2.2 mathematically. After that, we will implement the equations into the programming language julia in sec. 4. Using the extension “QuantumOptics.jl” developed in the Ritsch CQED-group at the Institute for Theoretical Physics of the University of Innsbruck, we can concentrate on the underlying physics, as we will not have to worry about mathematical details solving the equations.

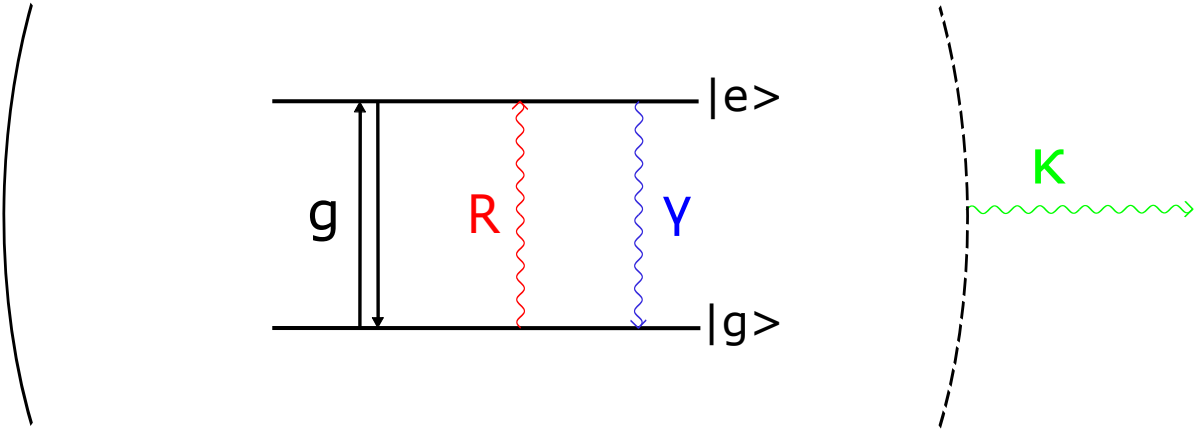


Figure 2.2: Complete model of the laser. Depicted is a level system with ground state $|g\rangle$ and excited state $|e\rangle$. The two-level-system interacts with the light field inside the cavity with a rate g , depicted by the black arrows. R represents the pump rate from the ground state $|g\rangle$ to the excited state $|e\rangle$, γ stands for the rate of spontaneous emission. The two-level-system is located inside a cavity, which is confined by a fully reflecting mirror on its right side and a partially reflecting mirror (dashed line) on its left side. Photons, that get emitted into the cavity by the atoms, can leak out of the cavity with a rate of κ .

3 Derivation of the model

3.1 The Jaynes Cummings-Model

The Jaynes Cummings-Model is a common model to describe an atom interacting with a surrounding monochromatic field mode of a cavity. It consists of three parts:

- the Hamiltonian for the atom \hat{H}_A
- the Hamiltonian for the field \hat{H}_F
- the Hamiltonian for the interaction \hat{H}_{AF}

The goal of the following sections is to derive these parts, then the final Jaynes-Cummings-Hamiltonian is given by

$$\hat{H}_{JC} = \hat{H}_A + \hat{H}_F + \hat{H}_{AF}. \quad (3.1)$$

The following derivations follow [7], for more details one may have a look into this source.

3.1.1 The Atom-Hamiltonian \hat{H}_A

A two-level-system can be mathematically described by a spin-1/2-system in a magnetic field \mathbf{B} . Without loss of generality, we assume that the magnetic field is oriented in the z-direction, then the hamilton operator is proportional to $\hat{\sigma}_z$: $\hat{H} = \alpha \cdot \hat{\sigma}_z$ with α as a proportionality constant. As a complete and orthonormal basis we use the eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of the σ_z -operator. Their eigenvalues are +1 and -1 respectively. By defining

$$\begin{aligned} |\uparrow\rangle &\equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |\downarrow\rangle &\equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

we can use the third pauli-Matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as the matrix representation of $\hat{\sigma}_z$. For convenience we set the level of zero energy in the middle between the upper and lower level. We can see, that the proportionality constant in the Hamilton operator must be $\frac{\Delta E}{2}$, when ΔE denotes the energy difference between upper and lower level:

$$\langle \uparrow | \hat{H} | \uparrow \rangle - \langle \downarrow | \hat{H} | \downarrow \rangle = \alpha \cdot (1 - (-1)) = \alpha \cdot 2 \stackrel{!}{=} \Delta E. \quad (3.2)$$

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Switching from a spin-1/2-system to the two-level-system of an atom, the state $|\uparrow\rangle$ corresponds to the excited state $|e\rangle$ and the state $|\downarrow\rangle$ corresponds to the ground state $|g\rangle$. The energy difference ΔE equals $\hbar\omega_a$, where ω_a is the frequency of the transition between excited state and ground state of the atom. In conclusion, the hamilton operator of the atom reads

$$\hat{H}_A = \frac{\hbar\omega_a}{2}\hat{\sigma}_z \quad (3.3)$$

For the later use we also define $\hat{\sigma}_+$ and $\hat{\sigma}_-$ by $\hat{\sigma}_{\pm} = \frac{1}{2}(\hat{\sigma}_x \pm i\hat{\sigma}_y)$ [6]. Using the matrix representation of $\hat{\sigma}_x$ and $\hat{\sigma}_y$ [7] we arrive at

$$\hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (3.4)$$

3.1.2 The Field-Hamiltonian \hat{H}_F

For the derivation of the field-hamiltonian \hat{H}_F we start from the Maxwell equations in vacuum, which means without electric charges ρ and electric currents \mathbf{j} [5]:

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = 0 \quad (3.5)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \quad (3.6)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}}{\partial t} \quad (3.7)$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (3.8)$$

$$(3.9)$$

Here, $\mathbf{E}(\mathbf{r}, t)$ denotes the electric field at the point \mathbf{r} in space at time t , $\mathbf{B}(\mathbf{r}, t)$ denotes the magnetic field and μ_0 and ϵ_0 are the vacuum permeability and permittivity respectively. We introduce the vector potential $\mathbf{A}(\mathbf{r}, t)$ and the scalar potential $\phi(\mathbf{r}, t)$ with the definitions [7]

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t) \quad (3.10)$$

$$-\nabla\phi = \mathbf{E}(\mathbf{r}, t) + \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad (3.11)$$

As we could always add a gradient-field to $\mathbf{A}(\mathbf{r}, t)$ in eq. 3.10 and this would not change the magnetic field $\mathbf{B}(\mathbf{r}, t)$, we have a certain gauge freedom. In this case we choose a gauge function in a way, that $\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0$, this is known as the Coulomb gauge [7]. We can also see, because of $\rho(\mathbf{r}, t) = 0$, that $\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = 0$. By simple calculations we can see, that eq. 3.6 is trivially fulfilled. As we assume, that the fields are sufficiently smooth, that we can interchange spatial differentiations with differentiations

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with respect to time, so it is also easy to see, that eq. 3.7 is fulfilled. Inserting eq. 3.10 into eq. 3.8 and using eq. 3.11 to rewrite the electric field $\mathbf{E}(\mathbf{r}, t)$ in terms of the vector potential $\mathbf{A}(\mathbf{r}, t)$ yields a wave equation for the vector potential:

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{A}(\mathbf{r}, t) = 0 \quad (3.12)$$

Here, we have used, that $\nabla \times \nabla \times \mathbf{A}(\mathbf{r}, t) = \nabla(\nabla \cdot \mathbf{A}(\mathbf{r}, t)) - \nabla^2 \mathbf{A}(\mathbf{r}, t) = -\nabla^2 \mathbf{A}(\mathbf{r}, t)$ and the fact, that $\mu_0 \epsilon_0 = \frac{1}{c^2}$, where c is the speed of light.

We will only look at one space dimension and without loss of generality we will assume the fields to be position-dependent in the z -direction. We will also assume, that the vector potential $\mathbf{A}(\mathbf{r}, t)$ and in consequence also the electric field $\mathbf{E}(\mathbf{r}, t)$ points in the x -direction. By eq. 3.10 this will imply, that the magnetic field $\mathbf{B}(\mathbf{r}, t)$ points in the y -direction. Using the separation ansatz $\mathbf{A}(z, t) = g(z) \cdot \xi(t) \cdot \mathbf{e}_x$ for eq. 3.12, applying the boundary condition $\mathbf{E}(z = 0, t) = \mathbf{E}(z = L, t) = 0$ and only considering the lowest mode we get the solution

$$\mathbf{A}(\mathbf{r}, t) = A(z) \cdot \mathbf{e}_x = C \cdot \sin(kz) \cdot \xi(t) \cdot \mathbf{e}_x, \quad \ddot{\xi}(t) = -\omega_c^2 \cdot \xi(t), \quad (3.13)$$

where $k = \omega_c \cdot c$. With eq. 3.10 and 3.11 we can calculate the magnetic and electric field respectively, using $\ddot{\xi}(t) = -\omega_c^2 \cdot \xi(t)$ from the separation ansatz and defining $q(t) = \dot{\xi}(t)$ ($q(t)$ has now dimensions of length [7]) yields

$$\mathbf{E}(z, t) = -\frac{\partial \mathbf{A}(z, t)}{\partial t} = -C \sin(kz) q(t) \mathbf{e}_x \quad (3.14)$$

$$\mathbf{B}(z, t) = \nabla \times \mathbf{A}(z, t) = C \cos(kz) \frac{k}{-\omega_c^2} \dot{q}(t) \mathbf{e}_y \quad (3.15)$$

The electromagnetic energy of the fields is given by [5]

$$H_{\text{em}} = \frac{1}{2} \int_V dV \left(\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2 \right), \quad (3.16)$$

in our case V denotes the volume of the cavity. Inserting eq. 3.14 and 3.15 into 3.16 and using $\int_0^L \sin^2(kz) dz = \int_0^L \cos^2(kz) dz = L/2$ we arrive at

$$H_{\text{em}} = \frac{VC^2}{4} \epsilon_0 \cdot \left(q(t)^2 + \frac{1}{\omega_c^2} \dot{q}(t)^2 \right). \quad (3.17)$$

This looks a lot like a harmonic oscillator with the hamilton function of a harmonic oscillator

$$H = \frac{1}{2} m \omega_c^2 \left(q(t) + \frac{1}{\omega_c^2} \frac{p(t)^2}{m^2} \right) \quad (3.18)$$

with $p(t) = m \dot{q}(t)$. Comparing yields $m = \frac{VC^2}{2\omega_c^2} \epsilon_0$. We know how to canonically quantize the harmonic oscillator: Defining the annihilation/lowering operator \hat{a} and creation/rais-

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ing operator \hat{a}^\dagger by [2]

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega_c m}} \cdot (\hat{a}^\dagger + \hat{a}) \quad (3.19)$$

$$\hat{p} = i\sqrt{\frac{m\omega_c \hbar}{2}} \cdot (\hat{a}^\dagger - \hat{a}) \quad (3.20)$$

and inserting this into eq. 3.18 we arrive at the known hamilton operator of the quantum harmonic oscillator

$$\hat{H} = \hbar\omega_c(\hat{a}^\dagger\hat{a} + \hat{1}/2). \quad (3.21)$$

As we are free to add a constant in the expression for the hamilton operator (this corresponds to shift the level of zero energy), we can leave out the $\hat{1}$ in eq. 3.18 and arrive at the final hamilton operator for the electric field:

$$\hat{H}_F = \hbar\omega_c\hat{a}^\dagger\hat{a}. \quad (3.22)$$

Using the operator expressions for $q(t)$ and $p(t)$ (eq. 3.19 and 3.20), we can also express the electric field and the magnetic field by operators:

$$\hat{\mathbf{E}} = E_0 \cdot \sin(kz) \cdot (\hat{a}^\dagger + \hat{a}) \quad (3.23)$$

$$\hat{\mathbf{B}} = -B_0 \cdot \cos(kz) \cdot i(\hat{a}^\dagger - \hat{a}) \quad (3.24)$$

Here, $E_0 = \sqrt{\frac{\hbar\omega_c}{V\epsilon_0}}$ and $B_0 = \frac{1}{c}\sqrt{\frac{\hbar\omega_c}{V\epsilon_0}}$, this can be calculated by using $m = \frac{VC^2}{2\omega_c^2}\epsilon_0$ from the comparison between the electromagnetic energy and the hamilton function of the quantum harmonic oscillator.

3.1.3 The Interaction-Hamiltonian \hat{H}_{AF}

Now we want derive the hamiltonian, which describes the interaction between the light field and the atoms. Classically, the interaction energy U is given by [3]

$$U = -\mathbf{d} \cdot \mathbf{E}(\mathbf{r}, t). \quad (3.25)$$

For canonical quantisation, we have to replace the electric dipolemoment \mathbf{d} and the electric field \mathbf{E} by their operator representation. We already derived the operator representation of the electric field \mathbf{E} in the previous section, now we are focusing on the operator representation of the dipole moment. The dipole operator is given by

$$\hat{\mathbf{d}} = q \cdot \hat{\mathbf{r}}, \quad (3.26)$$

where q is the charge and $\hat{\mathbf{r}}$ is the position operator. Choosing the two levels of the atom as a complete and orthonormal basis, we can represent the dipole operator as a matrix, also we are only interested in the component of the dipole operator in a certain direction \mathbf{e} , the direction of the electric field. The ij -th matrix element d_{ij} is then given by

$$d_{ij} = \mathbf{e} \cdot \langle i|\hat{\mathbf{d}}|j\rangle = \mathbf{e} \cdot q \int d\mathbf{r}^3 \cdot \psi_i^* \mathbf{r} \psi_j, \quad (3.27)$$

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When i is equal to j , the integrand simplifies to $\mathbf{r} \cdot |\psi_i|^2$. This integrand is odd with respect to the position \mathbf{r} , and an odd integrand integrated over a symmetrical region yields zero (assuming convergence of the integral), therefore the diagonal elements must be zero. By the definition in eq. 3.27 we can see, that the two off diagonal elements of the matrix representation of the dipole operator are the complex conjugate of each other. Without loss of generality, we assume, that the elements are real, as we are allowed to multiply both atom wavefunctions with a arbitrary, global phase in order to make the off-diagonal elements real. In conclusion, the (scalar) dipole operator in matrix representation is given by

$$\hat{d} = d \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = d (\hat{\sigma}_+ + \hat{\sigma}_-). \quad (3.28)$$

In the last step, we made use of the definition of $\hat{\sigma}_+$ and $\hat{\sigma}_-$ (compare sec. 3.1.1). We can now insert the matrix representations into eq. 3.25 and arrive at

$$\hat{H}_{\text{AF}} = \hbar g (\hat{\sigma}_+ + \hat{\sigma}_-) \cdot (\hat{a} + \hat{a}^\dagger), \quad g = -\frac{d E_0}{\hbar} \sin(kz) \quad (3.29)$$

Expanding the brackets yields

$$\hat{H}_{\text{AF}} = \hbar g (\hat{\sigma}_+ \hat{a} + \hat{\sigma}_+ \hat{a}^\dagger + \hat{\sigma}_- \hat{a} + \hat{\sigma}_- \hat{a}^\dagger) \quad (3.30)$$

$$\approx \hbar g (\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger) \quad (\text{RWA}), \quad (3.31)$$

where we applied the rotating-wave-approximation (RWA) in the last step. This approximation can be understood by the following considerations: Neglecting the interaction hamiltonian \hat{H}_{AF} , one can derive the time evolution of the operators in the heisenberg-picture [7]:

$$\begin{aligned} \hat{a}(t) &= \hat{a}(0) e^{-i\omega_c t} \\ \hat{a}^\dagger(t) &= \hat{a}^\dagger(0) e^{i\omega_c t} \\ \hat{\sigma}_\pm(t) &= \hat{\sigma}_\pm(0) e^{\pm i\omega_a t} \end{aligned}$$

We see, that ω_a and ω_c get added together in the terms for $\hat{\sigma}_+ \hat{a}^\dagger$ and $\hat{\sigma}_- \hat{a}$ and subtracted from each other in the terms for $\hat{\sigma}_+ \hat{a}$ and $\hat{\sigma}_- \hat{a}^\dagger$. As ω_a and ω_c are of approximately equal magnitude, the terms where these frequencies add rotate very quickly compared to the terms where these frequencies are subtracted from each other. Therefore, the fast-rotating terms are averaging out over time and we neglect them.

3.1.4 Dicke States

In our model we want to consider more than only one atom. However, the dimensions of the Hilbert space and in further consequence the computing time become big very

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quickly. For N two-level-atoms, the dimensions of the corresponding Hilbert space is 2^N . To reduce the dimensions of the Hilbert space, we assume that the atoms are indistinguishable. This leads to only considering symmetrical states, as interchanging the atoms should not have an impact on the wavefunction. Then the dimensions of the Hilbert space reduce to $N + 1$, which is equal to the number of energy levels. Similar to describing one two-level-system by a spin-1/2-system, we can describe N two-level-systems by a spin- $N/2$ -basis. We define the collective ladder operators S_{\pm} and the collective S_z -operator by summing over the corresponding operators for one two-level-system ([6] and [13]):

$$\hat{S}_{\pm} = \sum_{i=1}^N \hat{S}_{\pm}^{(i)}, \quad \hat{S}_z = \sum_{i=1}^N \hat{S}_z^{(i)} \quad (3.32)$$

We are allowed to assume indistinguishability of the atoms, because the initial state will be the ground state of all atoms, and therefore a symmetric state. Furthermore, also the time evolution does not change a symmetric state into a non-symmetric state [13], as there are only collective operators responsible for the time evolution. To illustrate this, we can look at the example of $N = 3$ atoms:

$$\begin{aligned} |E_0\rangle &= \hat{S}_+^0 |ggg\rangle = |ggg\rangle \\ |E_1\rangle &= \hat{S}_+ |ggg\rangle \propto |gge\rangle + |geg\rangle + |egg\rangle \\ |E_2\rangle &= \hat{S}_+^2 |ggg\rangle \propto |eeg\rangle + |ege\rangle + |gee\rangle \\ |E_3\rangle &= \hat{S}_+^3 |ggg\rangle \propto |eee\rangle \end{aligned}$$

With $N = 3$ atoms, we have $N + 1 = 4$ energy levels (0, 1, 2 and 3 atoms in the excited state). Applying the collective creation operator \hat{S}_+ to the ground state $|ggg\rangle$ once yields $|E_1\rangle = \hat{S}_+ |ggg\rangle \propto |gge\rangle + |geg\rangle + |egg\rangle$. Exchanging for example the first and second atom we get the same wavefunction again. This works similar for interchanging other atoms, for other states and also for higher N .

In conclusion, the full Jaynes-Cummings-Hamiltonian considering all terms reads

$$\hat{H}_{\text{JC}} = \hat{H}_A + \hat{H}_F + \hat{H}_{\text{AF}} \quad (3.33)$$

$$= \frac{\hbar\omega_a}{2} \hat{S}_z + \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar g (\hat{S}_+ \hat{a} + \hat{S}_- \hat{a}^\dagger). \quad (3.34)$$

This Hamiltonian describing multiple atoms is also called Tavis-Cummings-Hamiltonian [13].

3.2 Derivation of a prototyp master equation

In the previous section we derived the Jaynes-Cummings-Hamiltonian, which describes the atoms, the field inside the cavity and their interaction. Now we want to have a

3 Derivation of the model

look at the pump and decay rate and also the loss of photons from the cavity by the leaky mirror. So we want to derive equations of motion for systems interacting with a reservoir. In the case of the cavity with a certain light field, the light field interacts with the surroundings of the cavity through the mirrors. Here we are only interested in the time evolution of the cavity field, and not of the surrounding reservoir. Naturally, when looking at systems interacting with environments, the energy of the system does not need to be conserved, as system and environment can exchange energy. This is why there is no Schrödinger equation for the system disregarding the environment. To describe this sort of processes, we incorporate methods of statistical physics. Then, the pure states $|\psi\rangle$ become mixed states described by a density operator ρ and the Schrödinger equation becomes the Liouville equation [12]:

$$\frac{d\rho}{dt} = \frac{i}{\hbar} \cdot [\rho, \hat{H}] \quad (3.35)$$

This equation is can be derived easily, if one recalls that the density operator can be written as $\rho = \sum_i p_i \cdot |\psi_i\rangle\langle\psi_i|$, where p_i is the probability to be in the state $|\psi_i\rangle$ and these states satisfy the schrödinger equation. Differentiating ρ with respect to time using product rule and inserting the schrödinger equation yields eq. 3.35. However, this equation is only true when looking at the system and the environment.

To get an equation of motion only for the system itself, we will trace eq. 3.35 over the reservoir degrees of freedom. The resulting differential equation is known as a master equation. In this section we will derive the master equation for a system modelled as a harmonic oscillator interacting with a reservoir, which can be modelled as a sum of harmonic oscillators. The reservoir could for example be the vacuum, in which the the electromagnetic modes at a certain temperature can be described by a harmonic oscillator. From this prototype master equation we will be able to deduce the corresponding terms for our model of the atoms inside a cavity, which is interacting with an environment.

As already stated, we assume, that our system can be described by a harmonic oscillator and the surrounding reservoir is modelled as a sum of harmonic oscillators [12]:

$$H_{0,A} = \hbar\omega c^\dagger c, \quad [c, c^\dagger] = 1, \quad \text{system} \quad (3.36)$$

$$H_{0,B} = \sum_j \hbar\omega_j b_j^\dagger b_j, \quad [b_j, b_k^\dagger] = \delta_{jk}, \quad \text{reservoir} \quad (3.37)$$

Moreover, we assume the hamilton describing the type of interaction being $\hat{H}_{\text{int}} = \sum_j g_j (c^\dagger b_j + c b_j^\dagger)$ with g_j being coupling constants, this leads to the full Hamiltonian being

$$H = \underbrace{\hbar\omega c^\dagger c + \sum_j \hbar\omega_j b_j^\dagger b_j}_{:\hat{H}_0} + \underbrace{\sum_j g_j (c^\dagger b_j + c b_j^\dagger)}_{\hat{H}_{\text{int}}}. \quad (3.38)$$

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For the following derivations, it is convenient to work in the interaction picture, where an operator \hat{A} is defined by

$$\tilde{\hat{A}} = \exp \frac{i}{\hbar} \hat{H}_0 t \cdot \hat{A} \cdot \exp -\frac{i}{\hbar} \hat{H}_0 t, \quad (3.39)$$

where $\tilde{\hat{A}}$ is the operator in the interaction picture. Employing this definition for the density operator describing the system and the reservoir ρ_{AB} and differentiating with respect to time using product rule yields [12]

$$\frac{d\rho_{AB}}{dt} = -\frac{i}{\hbar} [\tilde{H}_{\text{int}}, \rho_{AB}]. \quad (3.40)$$

We can also write \tilde{H}_{int} as [12]

$$\tilde{H}_{\text{int}} = \hbar(G(t)c^\dagger + G^\dagger(t)c) \quad (3.41)$$

$$G(t) \equiv \sum_j g_j b_j \exp i(\omega - \omega_j)t. \quad (3.42)$$

Formally integrating the Liouville equation yields

$$\tilde{\rho}_{AB}(t) = \tilde{\rho}_{AB}(0) + \frac{1}{i\hbar} \int_0^t [\tilde{H}_{\text{int}}(t'), \tilde{\rho}_{AB}(t')] dt'. \quad (3.43)$$

Substituting this back into eq. 3.40 and tracing over the reservoir variables yields [12]

$$\frac{\partial \tilde{\rho}_A}{\partial t} = -\frac{1}{\hbar^2} \cdot \int_0^t \text{Tr}_B \{ [\tilde{H}_{\text{int}}(t), [\tilde{H}_{\text{int}}(t'), \tilde{\rho}_{AB}]] \} dt'. \quad (3.44)$$

Here we used, that

$$\text{Tr}_B[\tilde{H}_{\text{int}}, \tilde{\rho}_{AB}(0)] = 0, \quad (3.45)$$

which can be shown by assuming no correlation at $t = 0$ between the system and the reservoir, and $\tilde{\rho}_B$ being the density operator for a thermal bath of Temperature T in canonical distribution. More details on the derivations are given in [12]. Now we need to evaluate the double commutator in eq. 3.44 and integrate to get the final result. As this is a rather long calculation, we will not have a deeper look into it. However, if one wishes to comprehend the derivations, [9] is a detailed source to look at. An important step is to assume, that there is no correlation between bath and reservoir, so the full density operator can be written as

$$\tilde{\rho}_{AB}(t) = \tilde{\rho}_A(t) \otimes \tilde{\rho}_B(0), \quad (\text{Markovian assumption}). \quad (3.46)$$

This is known as the Markovian assumption [12]. One can also see, that the state of the reservoir $\tilde{\rho}_B$ is assumed to not change over time, so the change of the system does not affect the reservoir. This is justified by making the reservoir a lot larger than the system.

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Furthermore, one has to assume, that the modes of the reservoir lie dense enough to replace the sum with an integral

$$\sum_j \rightarrow \int D(\omega_j) d\omega_j. \quad (3.47)$$

The final equation of motion for the system, the master equation, reads as follows:

$$\dot{\tilde{\rho}}_A = \epsilon (1 + \bar{n}) (2c\tilde{\rho}c^\dagger - c^\dagger c\tilde{\rho} - \tilde{\rho}c^\dagger c) + \quad (3.48)$$

$$+ \epsilon \cdot \bar{n} \cdot (2c^\dagger \tilde{\rho} c - cc^\dagger \tilde{\rho} - \tilde{\rho}cc^\dagger). \quad (3.49)$$

This equation describes a system with raising/lowering operators c/c^\dagger , interacting with a thermal reservoir with average photon number \bar{n} at a certain coupling rate ϵ with dimension 1/time. The first term, $\epsilon (1 + \bar{n}) (2c\tilde{\rho}c^\dagger - c^\dagger c\tilde{\rho} - \tilde{\rho}c^\dagger c)$, describes the process that the system changes its state from a higher state to a lower state [13], this happens with a rate of $\epsilon (1 + \bar{n})$. Analogously, the second term, $\epsilon \cdot \bar{n} \cdot (2c^\dagger \tilde{\rho} c - cc^\dagger \tilde{\rho} - \tilde{\rho}cc^\dagger)$, describes the process that the system changes its state from a lower state to a higher state, this happens with a rate of $\epsilon \cdot \bar{n}$. This can be seen by setting $\bar{n} = 0$, which corresponds to a temperature of 0 K. Then only the first term survives, and this term obviously must be the term describing the dissipation (compare [13]). For a thermal reservoir, these two rates are coupled.

3.3 The differential equation for the laser

To get to the final equation of motion for our model (see fig. 3.1), we have to incorporate two systems interacting with their surroundings:

- the electromagnetic field inside the cavity, which loses photons to the surrounding
- the atoms, which interact with the inside of the cavity by emitting photons into the cavity

First, we have a look at the cavity losing photons to its surrounding. From the prototype master equation we can deduce the following: the system under consideration is the cavity, more precisely the electromagnetic field with the lowering/raising operators \hat{a}/\hat{a}^\dagger , these operators correspond to the operators \hat{c}/\hat{c}^\dagger in the prototype master equation. We assume, that the environment does not excite the electric field inside the cavity, which corresponds to $\bar{n} = 0$ in the prototype master equation. Finally, the rate at which the photons leak from the cavity is given by κ (compare fig. 3.1). Altogether this leads to the following term of so-called Lindblad-form :

$$L_\kappa[\rho] = \kappa (2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \quad (3.50)$$

This term describes the interaction between the electromagnetic field inside the cavity with the surrounding modes outside the cavity.

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Next, we extend the system of the atoms by adding an incoherent pump with rate R and allowing the atoms to incoherently emit photons outside with the rate γ (compare fig. 3.1). The system of the atoms has the lowering/raising operators S_-/S_+ , these correspond to the \hat{c}/\hat{c}^\dagger - operators of the prototype master equation. The surrounding is not a thermal reservoir like in the prototype master equation, where the transition rates are coupled. Now the rate of transition into the higher state is given by the pump rate R and the rate of transition into the lower state is given by the decay rate γ , so they decouple, which is again necessary to achieve population inversion. In conclusion, we get two more terms of Lindblad-form

$$L_\gamma[\rho] = \frac{\gamma}{2} (2S_- \rho S_+ - S_+ S_- \rho - \rho S_- S_+) \quad (3.51)$$

$$L_R[\rho] = \frac{R}{2} (2S_+ \rho S_- - S_- S_+ \rho - \rho S_+ S_-) \quad (3.52)$$

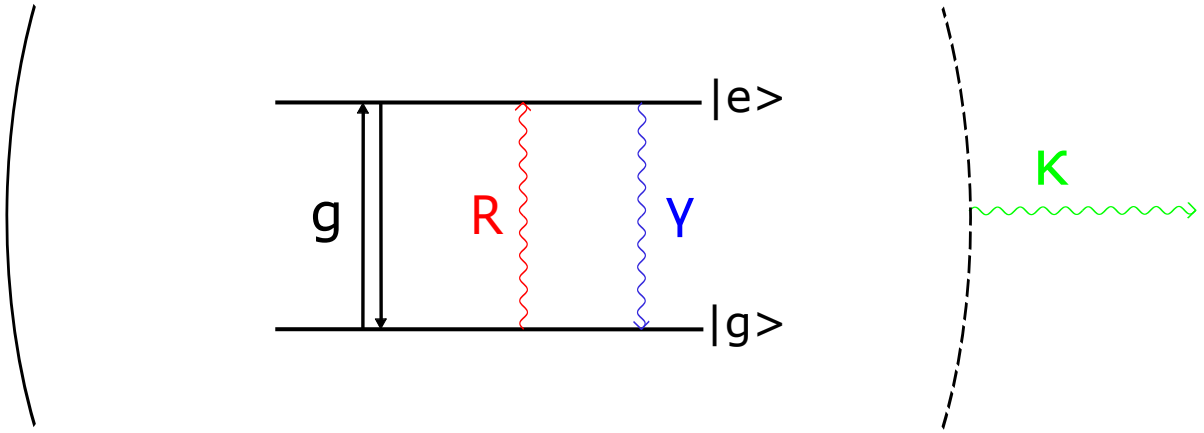


Figure 3.1: Complete model of the laser. Depicted is the two level system with ground state $|g\rangle$ and excited state $|e\rangle$. The two-level-system interacts with the light field inside the cavity with a rate g , depicted by the black arrows. R represents the pump rate from the ground state $|g\rangle$ to the excited state $|e\rangle$, γ stands for the spontaneous emission. The two-level-system is located inside a cavity, which is confined by a fully reflecting mirror on its right side and a partially reflecting mirror (dashed line) on its left side. Photons, that get emitted into the cavity by the atoms, can leak out of the cavity with a rate of κ .

By adding all the terms in Lindblad-form describing the systems interacting with their environment, we arrive at

$$\begin{aligned} \dot{\rho} = & \frac{i}{\hbar} [\rho, H_{JC}] + \kappa (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) + \\ & + \frac{\gamma}{2} (2S_- \rho S_+ - S_+ S_- \rho - \rho S_- S_+) + \\ & + \frac{R}{2} (2S_+ \rho S_- - S_- S_+ \rho - \rho S_+ S_-) \end{aligned}$$

3 Derivation of the model

Here, ρ denotes the density operator of the atoms and the field, and H_{JC} denotes the Jaynes-Cummings-Hamiltonian. We get the additional $\frac{i}{\hbar}[\rho, H_{\text{JC}}]$ by changing from the interaction picture back into the schrödinger picture.

Now we have derived the equation of motion, according to which our system develops. In the next chapter we will implement this equation into a programming language and analyse the results.

4 Quantum dynamics of a single mode superradiant laser

4.1 The program

Now we want to solve the differential equation for our model and discuss its properties. This will be done numerically. For this we use the programming language julia, as for julia there is an extension called “Quantum Optics.jl” developed by the Ritsch CQED group of University of Innsbruck. In this toolbox, there are many quantum operators predefined, so one can do quantum optical calculations very conveniently. One also does not need to worry about approximations when solving differential equations like the schrödinger equation or a master equation. An extensive documentation including a lot of examples is given at <https://qojulia.org/documentation/>.

To implement our laser model, we use the following code (partially taken over from: <https://qojulia.org/documentation/examples/jaynes-cummings.html>) First, we need to include all necessary packages. We need “QuantumOptics” for the predefined operators and equations, “PyPlot” for drawing graphics, and “Linear Algebra” for some calculations:

```
using QuantumOptics
using PyPlot
using LinearAlgebra
```

Now we can define the parameters of our system and the bases. The command “FockBasis(N_{cutoff})” creates a basis for the state of electromagnetic field with the base vectors $|n\rangle$ being the state with n Photons. Together, they form a orthogonal basis of the Hilbert space [6]. Note, that because of the N_{cutoff} -parameter, which limits the number of photons in the cavity to N_{cutoff} , this basis is not complete and we will have to make sure that the state can be sufficiently described by this reduced set of base vectors. A brief discussion of this is given in sec. 4.4.

```
# parameter
N_cutoff=10
wc=0.1
wa=0.1
g=12;
```

#bases

```
b_fock=FockBasis(N_cutoff)
N_atoms=6
b_spin=SpinBasis(N_atoms//2)
b=b_fock ⊗ b_spin;
```

Next, we define the fundamental operators acting on the bases and the Jaynes-Cummings-Hamiltonian. One has to carefully consider, in which Hilbert space the operators are defined. Here, the first subspace corresponds to the fock basis describing the electromagnetic field, the second subspace describes the state of the atoms.

#fundamental operators

```
a = destroy(b_fock)
at = create(b_fock)
n = number(b_fock)

sm = sigmam(b_spin)
sp = sigmap(b_spin)
sz = sigmaz(b_spin);
```

Jaynes-Cummings-Hamiltonian

```
Hatom = wa*sz/2
Hfield = wc*n
Hint = g*(at ⊗ sm + a ⊗ sp)
H = one(b_fock) ⊗ Hatom + Hfield ⊗ one(b_spin) + Hint;
```

We define the initial state and the time interval, within the differential equation should be solved.

initial state

```
Ψ0 = fockstate(b_fock, 0) ⊗ spindown(b_spin)
```

time interval

```
T_end=1
dt=0.1
T = [0:dt:T_end];
```

Finally, we are ready to define our master equation and calculate the time evolution:

#master equation and time evolution

```
γ=1      #decay rate
R=20     #pump rate
κ=40     #rate of loss of photons from the cavity
J=[sqrt(γ)*one(b_fock) ⊗ sm,
   sqrt(R)*one(b_fock) ⊗ sp,
   sqrt(κ)*a ⊗ one(b_spin)];
tout, pt = timeevolution.master(T, Ψ0, H, J);    #time evolution
exp_n_master= real(expect(n ⊗ one(b_spin), pt));
exp_sz_master = real(expect(one(b_fock) ⊗ sz, pt));
```

4.2 Time evolution

In fig. 4.1 we show the time evolution of the average photon number $\langle n \rangle$ and the expectation value of the atomic population difference S_z for the parameter set of $N_{\text{atoms}}=6$, $g=10$, $\gamma=1$, $R=20$, $\kappa=40$. Both $\langle n \rangle$ and $\langle S_z \rangle$ start at their lowest possible values, as we

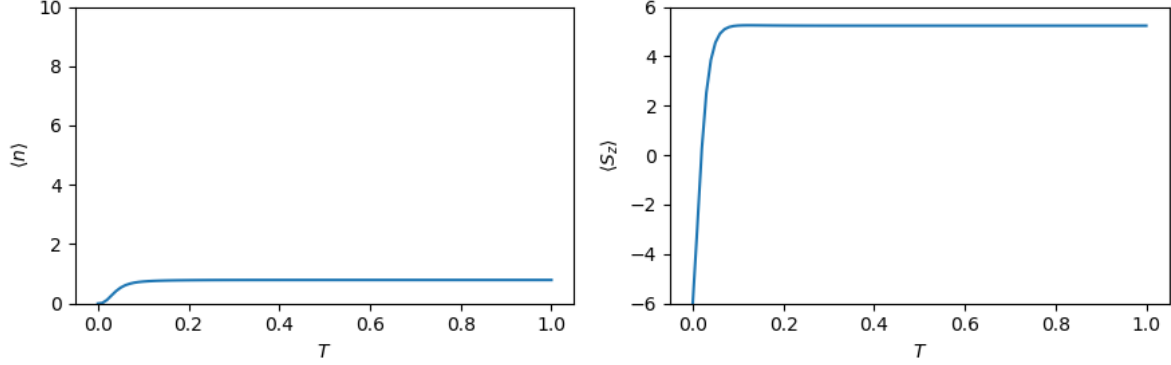


Figure 4.1: Time evolution of the photon number $\langle n \rangle$ and the S_z -operator with the parameters $N_{\text{atoms}} = 6$, $g=10$, $\gamma=1$, $R=20$, $\kappa=40$.

defined the ground state to be the initial state. We can see, that after an initial change, the system does not change significantly anymore, so it reaches its steady state. The average number of photons stays below 1, but still a lot of atoms are in the upper state (high $\langle S_z \rangle$). Clearly, population inversion has been achieved. If we are decreasing the pump rate R , then $\langle S_z \rangle$ also decreases to the point, where we do not have population inversion anymore.

4.3 The laser regime

To get an idea, for which parameters population inversion is reached, one can program a loop calculating the steady state and from that the expected photon number $\langle n \rangle$ and $\langle S_z \rangle$ indicating population inversion. This is depicted in fig. 4.2 for R going from 1 to 40. At the left subplot we can see, that population inversion is reached at an approximate pump rate of $R \approx 10$. The number of photons stays low, curiously it gets even lower for pump rates higher than ≈ 15 .

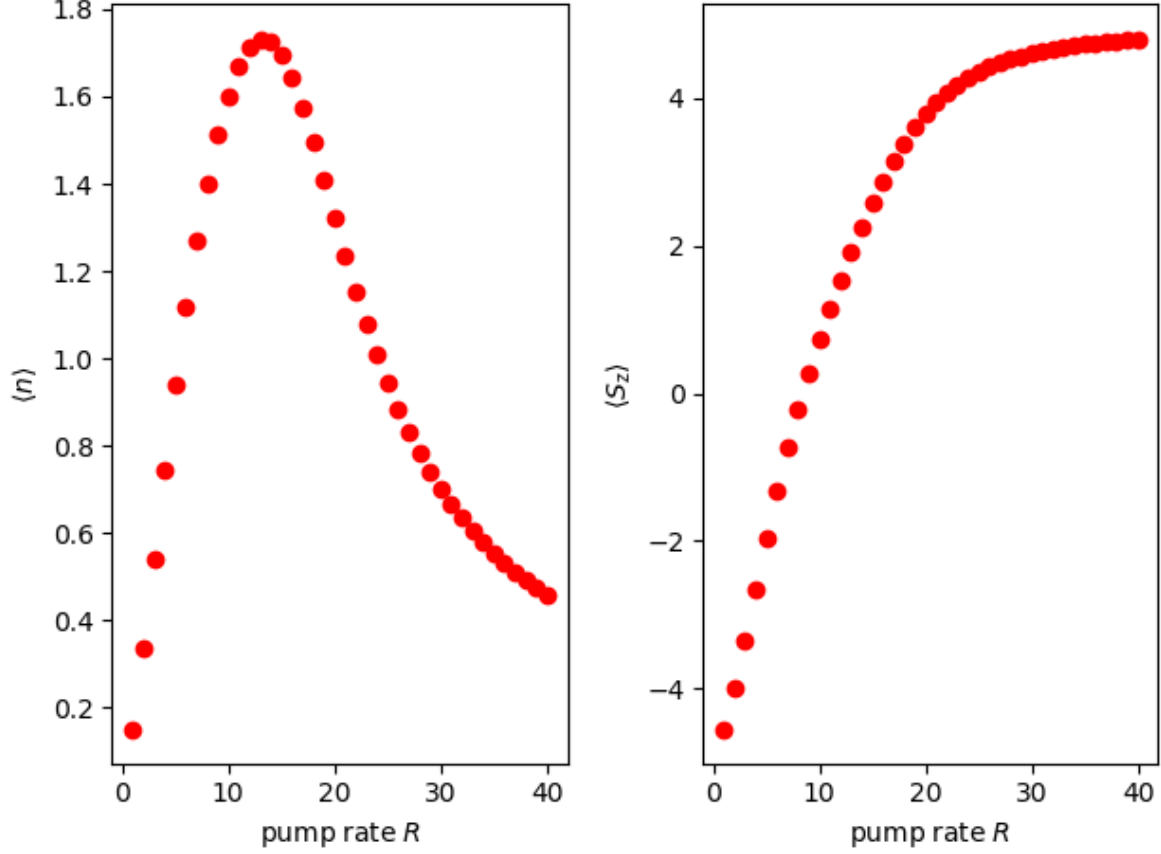


Figure 4.2: Expected photon number $\langle n \rangle$ and population inversion for various pump rates R . The other parameters are: $N_{\text{atoms}} = 5$, $g = 10$, $\gamma = 1$, $\kappa = 40$.

4.4 The photon number distribution

Now we have a look at the photon number distribution in the cavity. From the program we get a density operator describing the state of the electromagnetic field and the state of the atoms. As we are only interested in the state of the electromagnetic field, we trace over the Hilbert space of the atoms. Then, the probability P_n to find n photons in the cavity is

$$P_n = \langle n | \text{Tr}_{\text{atoms}} \{ \rho \} | n \rangle = \langle n | \rho_{\text{EM-field}} | n \rangle. \quad (4.1)$$

As the density operator $\rho_{\text{EM-field}}$ is given in a matrix representation using the $|n\rangle$ -states as a basis, P_n corresponds to the diagonal elements. In fig. 4.3 the photon distribution for various N_{atoms} is depicted. The other parameters are set to: $g=10$, $R=20$, $\kappa=40$. We see that the lower the parameter N_{atoms} , the better the distribution follows a poisson distribution with the corresponding expectancy value $\langle n \rangle$. The poisson distribution itself

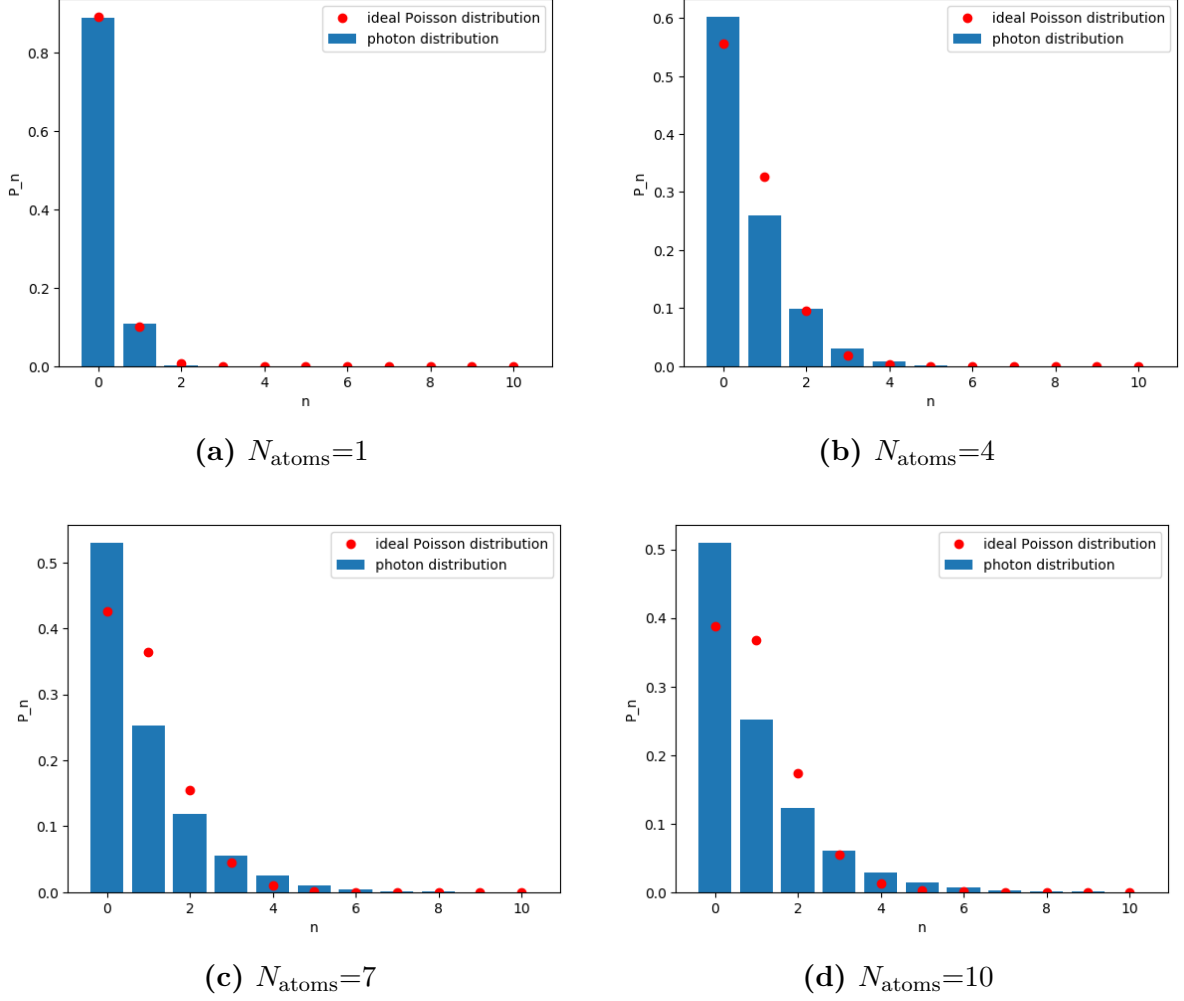


Figure 4.3: Photon number distribution inside the cavity for various N_{atoms} compared to an ideal Poisson-distribution. The other parameters are: $g=10$, $R=20$, $\kappa=40$.

is characteristic for coherent states [2], also called quasiclassical states. These states behave similar to classical states, for example, the uncertainty does not get increased with time and the expectancy values of position $\langle x \rangle$ and momentum $\langle p \rangle$ behave as classically expected.

Here we can also see the impact of the parameter N_{cutoff} . If we set the parameter at $N_{\text{cutoff}} = 5$, we would disregard the states with higher photon numbers, and therefore have an error in the calculation. However, the parameter should also not be too high, as this increases dimensions of the basis and therefore the computation time.

4.5 The spectrum and linewidth

One of the most important properties of a laser is its linewidth. To calculate the spectrum $S(\omega)$ and the linewidth, we make use of the Wiener-Chintschin-Theorem [10]:

$$S(\omega) = \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \cdot \langle a^\dagger(\tau) \cdot a(0) \rangle = \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \cdot g(\tau) \quad (4.2)$$

According to this theorem, the spectrum can be calculated by fourier-transforming the first order correlation function $g(\tau) := \langle a^\dagger(\tau) \cdot a(0) \rangle$. This is the quantum mechanical analog of the classical first order correlation function in electrodynamics.

To calculate the correlation function $g(\tau)$, we observe

$$\begin{aligned} g(\tau) &= \langle a^\dagger(\tau) a(0) \rangle = \text{Tr}_S \text{Tr}_B \{ a^\dagger(\tau) a(0) \rho \} = \\ &= \text{Tr}_S \text{Tr}_B \{ U^{-1}(\tau) a^\dagger(0) U a(0) \rho \} = \\ &= \text{Tr}_S \text{Tr}_B \{ U(\tau) U^{-1}(\tau) a^\dagger(0) U(\tau) \underbrace{a(0) \rho}_{:= \rho_{\text{new}}} U^{-1}(\tau) \} = \\ &= \text{Tr}_S \text{Tr}_B \{ a^\dagger(0) U(\tau) \rho_{\text{new}} U^{-1}(\tau) \} = \\ &= \text{Tr}_S (a^\dagger(0) \text{Tr}_B \{ U(\tau) \rho_{\text{new}} U^{-1}(\tau) \}) = \\ &= \text{Tr}_S \{ a^\dagger(0) \rho_{\text{new}}(\tau) \} \\ &= \langle a^\dagger(0) \rho_{\text{new}}(\tau) \rangle \end{aligned}$$

For the third line we used, that the trace is invariant under unitary transformations This suggests the following steps to calculate the spectrum [8]:

- define a new density operator $\rho_{\text{new}} := a\rho$, where ρ is the density operator of the system under consideration.
- calculate the time evolution of the new density operator $\rho_{\text{new}}(t)$ with the same master equation
- calculate the correlation function $g(\tau)$ by applying a^\dagger to the density operator $\rho_{\text{new}}(t)$ and tracing over the system variables.
- calculate the spectrum $S(\omega)$ by fourier transforming the correlation function $g(\tau)$

The program code for this looks as follows (partially taken over from: <https://qojulia.org/documentation/examples/correlation-spectrum.html>)

```
ρ_neu= (a ⊗ one(b_spin))*pt[end]
τ_end=10
τ=collect(range(0.0, stop=τ_end, length=2^14))
tout, ρ_neu_τ=timeevolution.master(τ, ρ_neu, H, J)
g_τ=expect(at ⊗ one(b_spin), ρ_neu_τ)
ω, spec = timecorrelations.correlation2spectrum(τ,
```

```
g_τ; normalize_spec=true);
plot(ω, spec)
xlabel(L"\omega")
ylabel("Intensity")
ylim(-0.1, 1.1)
xlim(-100, 100)
grid()
```

An example spectrum for the parameters: $N_{\text{atoms}} = 7$, $g = 10$, $\gamma = 1$, $R = 20$, $\kappa = 40$ is depicted in fig. 4.4. One has to choose the time steps for simulating quite small (2^{14} time bins in the above code example), otherwise the calculated spectrum has an offset.

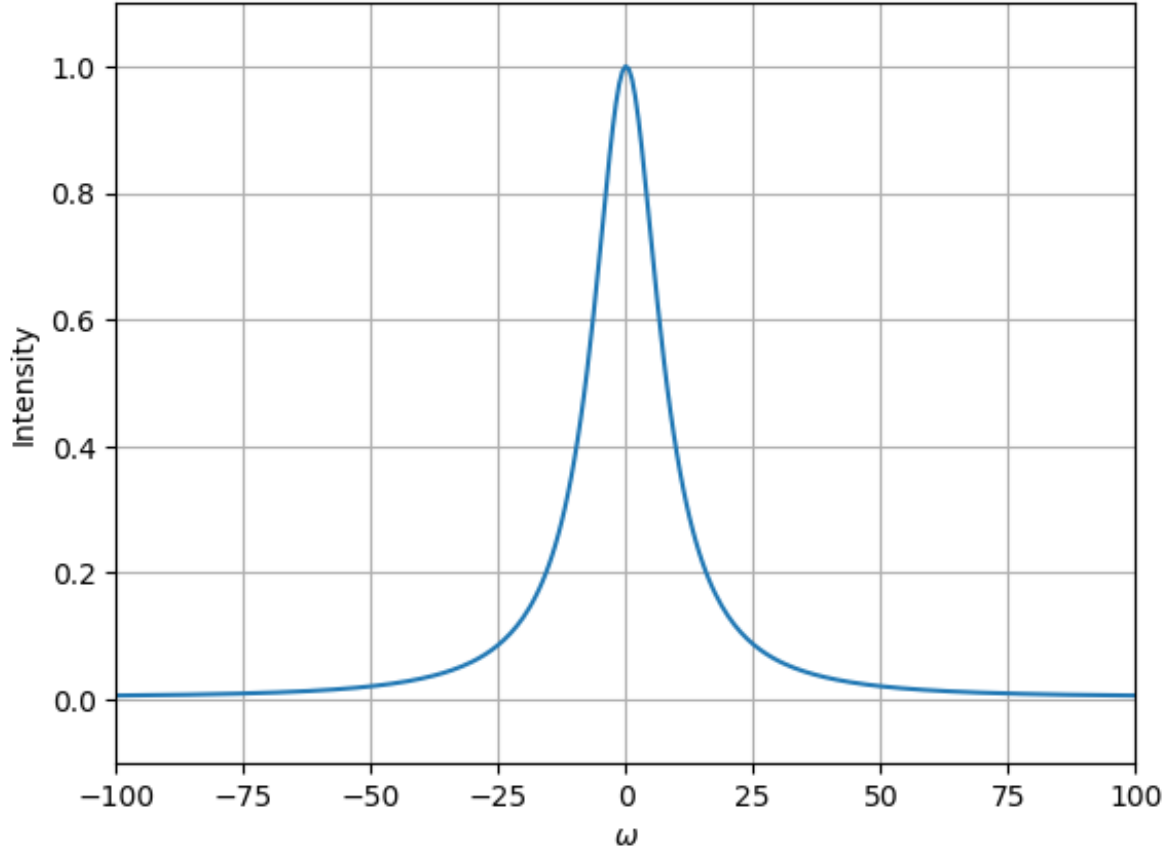


Figure 4.4: Normalised example spectrum. The parameters are: $N_{\text{atoms}} = 7$, $g = 10$, $\gamma = 1$, $R = 20$, $\kappa = 40$.

Now we want to study the influence of the number of atoms on the linewidth. Therefore we program a loop which calculates the spectrum for various N_{atoms} . From the spectrum we can extract the linewidth (Full-Width-Half-Maximum FWHM) for the various N_{atoms} . In fig. 4.5 the linewidth in dependency of N_{atoms} is depicted. The other parameters are:

$g = 10$, $\gamma = 2$, $R = 20$, $\kappa = 30$. We can clearly see, that for bigger N_{atoms} the linewidth gets smaller. However, for $N_{\text{atoms}} > 6$, the linewidth begins to increase again. This may

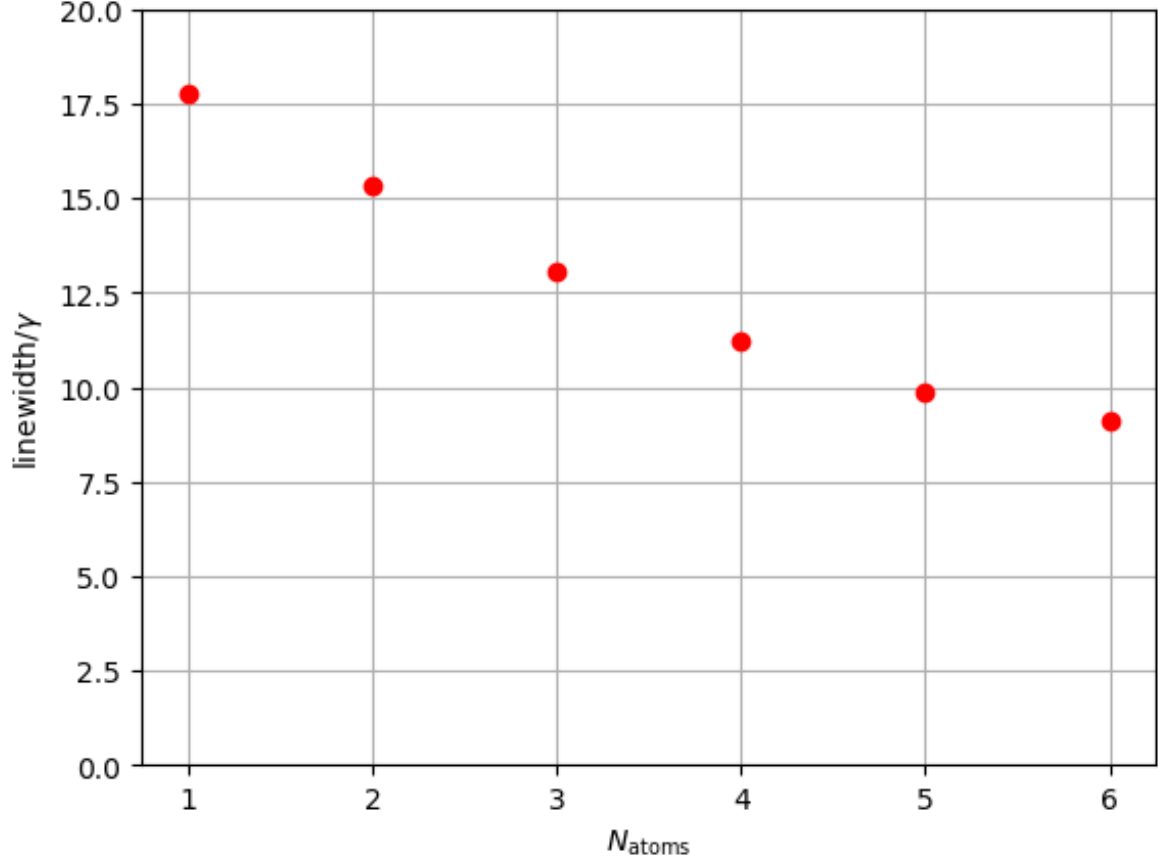


Figure 4.5: Linewidth with various N_{atoms} . The other parameters are: $g = 10$, $\gamma = 2$, $R = 20$, $\kappa = 30$.

be due to the fact, that with increasing N_{atoms} we leave the parameter regime in which the system works in laser mode.

4.6 The phase information

Finally, we have a look on the phase information of the states. Therefor we make use of the Husimi-function, also called Q function [5]. The Q-function is given by

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \rho_{\text{EM-field}} | \alpha \rangle \quad (4.3)$$

and gives a phase space probability distribution. Here, the coherent state $|\alpha\rangle$ denotes the eigenvector to the lowering operator \hat{a} with the complex eigenvalue α [2]. The Q-function is already implemented by the “QuantumOptics-Toolbox”, one simply has to

evaluate for different α and depict it for example in a contour-plot over a region of the complex plane. This is done in fig. 4.6 for increasing κ . On the x-axis the real part of α is plotted, the y-axis corresponds to the imaginary part of α . The other parameters are: $N_{\text{atoms}} = 5$, $g=10$, $R=20$. Yellow represents a high probability, blue represents a low probability.

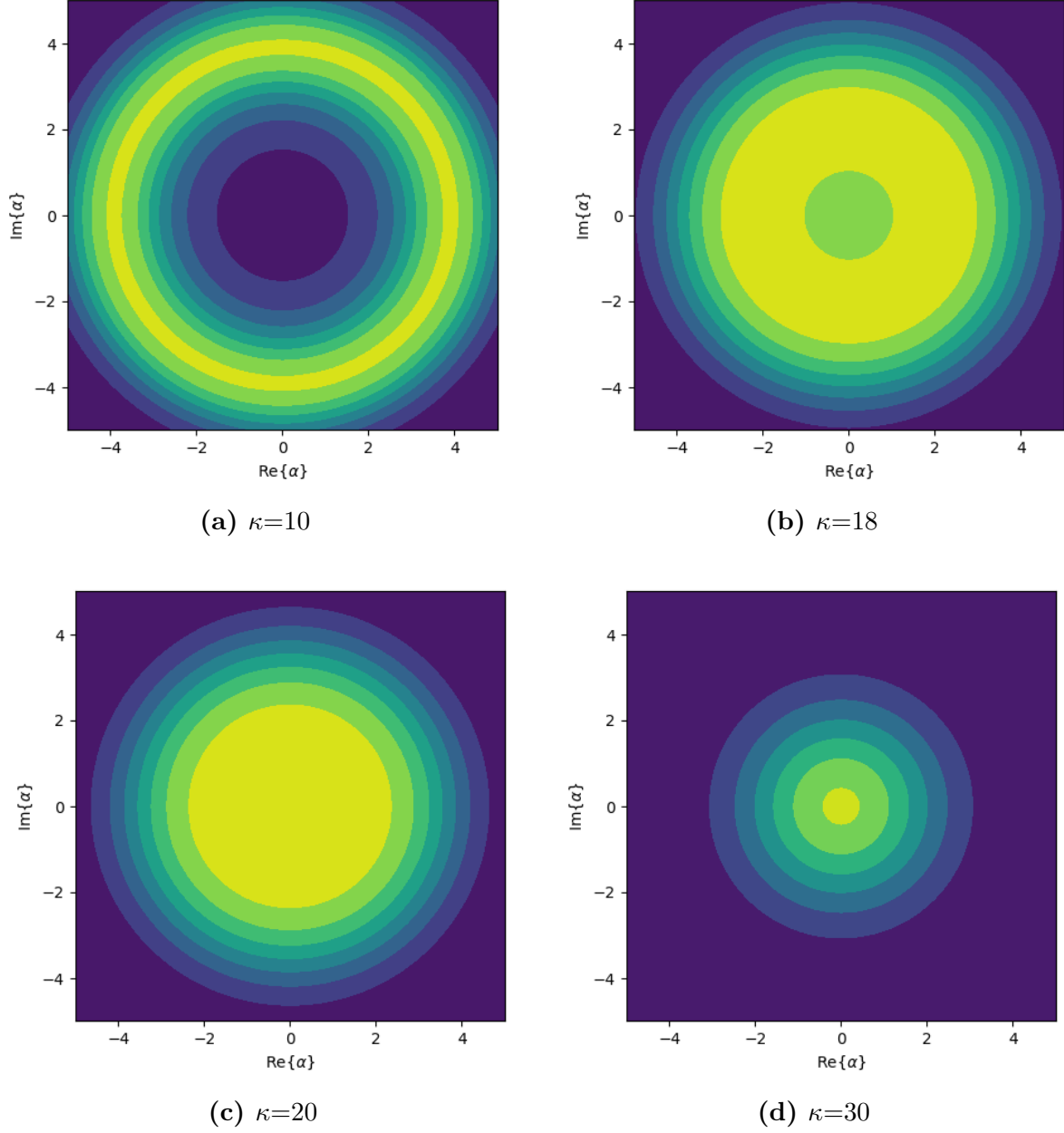


Figure 4.6: Husimi-Q-representation for increasing κ . The other parameters are: $N_{\text{atoms}} = 5$, $g=10$, $\gamma=1$ and $R=20$. On the x-axis there is the real part of α plotted, the y-axis corresponds to the imaginary part of α . Yellow represents a high probability, blue represents a low probability.

Theoretically, a coherent state has a gaussian distribution centered at α [5], whereas a number state $\rho = |n\rangle\langle n|$ appears more like a circle. We can see that for slowly entering the superradiant regime by increasing the photon loss rate κ , the distribution changes from a ring 4.6a to a distribution similar to gaussian 4.6d. In conclusion, the state of the electromagnetic field changes from a state similar to a number state to a state similar to vacuum state (= coherent state with eigenvalue $\alpha=0$) when entering the superradiant regime.

5 Conclusion

A model laser can be described by accounting for the interaction between atoms and field using the Jaynes-Cummings-Hamiltonian. The statistical processes of pumping and decay of the atoms and the leakage of photons from the cavity can be described by a master equation approach. This allowed us to derive the equation of motion for our model.

Using the programming language Julia and the extension “QuantumOptics.jl” developed by Univ.- Prof. Mag. Dr. Helmut Ritsch’s CQED group at the Institute for Theoretical Physics of the University of Innsbruck, we were able to calculate the time evolution and analyse the results. From the initial state, which was the ground state in our model, the system quickly reaches a steady state. The photon distribution follows a Poisson distribution, at least for a low number of atoms. This indicates, that the electromagnetic field is in a coherent state. Using the Wiener-Chintschin-Theorem we were able to extract the spectrum and in consequence the linewidth of the system. We have seen, that the linewidth decreases with increasing number of atoms. Using the Husimi-Q-representation for the phase space probability distribution, we could see that the state changes from a number state to a coherent state when increasing the leakage of photons and therefore entering the superradiant regime.

A next step would be to include the motion of the atoms into the discussion. For the application of cooling and trapping of dipole-interacting clock-atoms, for example, this has already been done in [8].

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