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Bachelor-Thesis

2D Solitons in QuantumOptics.jl

zur Erlangung des akademischen Grades
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vorgelegt von:

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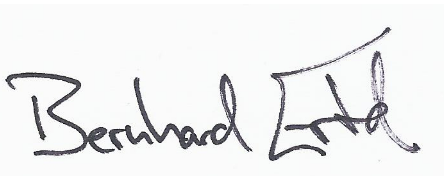

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Abstract

This thesis investigates 2D solitons modelled by the Gross-Pitaevskii-Equation using numerical methods in the QuantumOptics.jl toolbox. For better understanding the properties of Bose-Einstein-Condensate, Gross-Pitaevskii-Equation and its soliton solution are discussed shortly. Furthermore the QuantumOptics.jl framework is presented. The solitons were approximated by a Gaussian wave packet. Dynamic of a single soliton and collision of two solitons are studied for a negative coupling factor g. The soliton was found to be unstable; it either contracted or expanded. Two coupled solitons were stable for a longer time and the collision between two coupled solitons showed interference pattern.

Contents

1 Introduction to Bose-Einstein-Condensate

In a period of less than ten years the study of dilute quantum gases has changed from an esoteric topic to an integral part of contemporary physics, with strong ties to molecular, atomic, subatomic and condensed matter physics.[PS08]

[1] Nowadays the field of Bose-Einstein-Condensate (BEC) is a major topic of research. BEC is a state of matter that occurs when a dilute gas of bosons is cooled down to temperatures close to zero. The gas condenses and behaves as a macroscopic object but still shows quantum phenomena.

It lasts only 23 years back that BEC was accomplished for the very first time. 100 years ago nobody thought about this state of matter. It was Einstein to discuss it in 1924 for the first time. This is fascinating: theory was way ahead of experiments in this field of physics. Einstein's thoughts were based on Bose's paper [Bos24] about photon statistics and the Planck distribution. Einstein developed a similar theory for particles[Ein24]. The statistic that arose is called Bose-Einstein-Statistic. In the second treatise Einstein predicts the existence of what we today call Bose-Einstein-Condensate. However Einstein's theory was criticized by George Uhlenbeck. The criticism was mainly accepted and it kept Einstein and other physicists from discussing this state of matter any further.

Fritz London developed the idea of a "macroscopic wave function" for the theory of superconductivity starting in 1935. In 1937 London heard about Einstein's paper on BEC because Uhlenbeck had taken back his criticism. In the following Fritz London, Lazlo Tisza, Lew Landau and Nikolay Bogoliubov were dealing with superfluidity and superconductivity. These two concepts have strong analogies to BEC. London realized that and published a paper persisting on the connection between BEC and the superfluidity of liquid ^{4}He. This awoke interest in many physicist about BEC.

Griffin calls the years from 1957 to 1965 the "Golden Period": "*In this period, many important theorists attacked the interacting Bose-condensed gas problem. It was a hot topic during this period and, in my opinion, the final theoretical edifice is one of the great success stories in theoretical physics.*" [Gri99] This time brought many insights and a lot of literature about the "fictitious" Bose-condensed gas. Today the major equation describing BEC with atomic interaction is the Gross-Pitaevskii-Equation (GPE). It was derived by Gross and Pitaevskii independently in 1961.

At this point the level of technology was still not ready to experimentally show that such thing as BEC exists. But as time comes technology improves.

[1]For a more detailed report on the history I suggest to read [Gri99]. [Ket02] has a good description of the experimental evolution of BEC. [PS08] and [PS16] are comprehensive books explaining the phenomena of BEC and the theory of BEC.

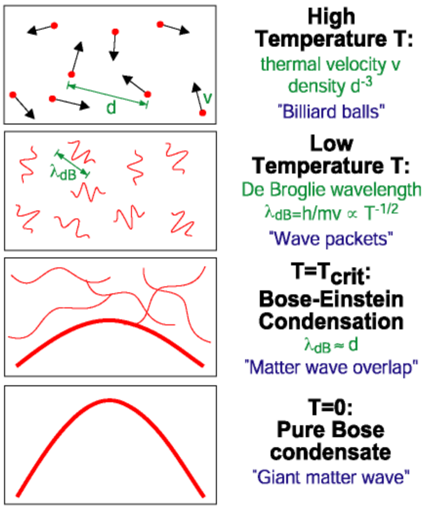

Figure 1: The concept of Bose-Einstein Condensation. (image from [Pro05])

Laser-cooling and evaporative cooling were developed and still are key concepts for reaching very low temperatures. Finally Wolfgang Ketterles group at MIT and Cornell and Wieman at JILA accomplished it for the first time. This was awarded with the nobel prize in 2001.
The first successful experiments were performed with rubidium and sodium atoms. Nowadays it is possible to make a condensate out of a list of atoms: ^{1}H, ^{7}Li, ^{23}Na, ^{39}K, ^{41}K, ^{52}Cr, ^{85}Rb, ^{133}Cs, ^{170}Yb, ^{4}He [PS08, in 2008] and probably a few more.

The following section gives a brief understanding of BEC. For further reading there are several comprehensive books.

1.1 Bose-Einstein Condensate

[PS08] Bose-Einstein Condensation is an extreme state of matter that occurs under the right conditions: A dilute gas of bosons has to be cooled down close to 0 K and the density has to be of the right dimensions. But lets have a look in more detail.
In quantum statistics particles are classified in two groups: fermions with an odd-half-spin $s = \frac{n}{2}$ with $n = \{1, 3, 5, ...\}$ and bosons with an integer spin $s = \{1, 2, 3, ...\}$.
Bosons obey the Bose-Einstein statistic: The mean number of occupation of a state ν at temperature T is given by

$$f^0(\epsilon_\nu) = \frac{1}{exp(\frac{\epsilon_\nu - \mu}{k_B T}) - 1}, \tag{1}$$

where ϵ_ν is the energy of the state ν, μ is the chemical potential, and k_B the Boltzmann factor. Fermions follow the Fermi-Dirac-statistic

$$f^0(\epsilon_\nu) = \frac{1}{exp(\frac{\epsilon_\nu - \mu}{k_B T}) + 1}. \tag{2}$$

The major difference between fermions and bosons regarding BEC is the occupation of quantum states. Fermions cannot occupy the same quantum state but bosons can. When two bosons occupy the same quantum state they are indistinguishable. This characteristic is important for Bose-Einstein-Condensate.
Assume a system of two identical bosons. The wave function for such a system is symmetric under interchange of those two bosons:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \Psi(\mathbf{r}_2, \mathbf{r}_1). \tag{3}$$

Same holds true for a larger number of atoms under the right conditions. When cooling down bosons the deBroglie wavelength $\lambda_{dB} = h/mv \propto T^{-1/2}$ of these bosons becomes larger and so the uncertainty of their position becomes larger. Another effect when cooling down bosons is that they start to occupy the same quantum state - the ground state. So we have a large number of bosons in the same quantum state. They are indistinguishable and delocalized. Meaning we can write the system of these particles as one macroscopic wave function

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_\mathrm{N}) = \Psi(\mathbf{r})^\mathrm{N}. \tag{4}$$

So we can treat this large number of particles as one particle. When this occurs it is called Bose-Einstein-Condensate. The mechanism is illustrated in figure 1.
This state of matter is fascinating because quantum phenomena can be observed on a macroscopic object. In experiments temperatures need to be of order 10^{-5} K or less. For that laser cooling is used followed by evaporative cooling. Evaporative cooling means removing atoms with higher energy and so cooling the remaining atoms.
The density of the gas is in the order of $10^{13} - 10^{15}\,\mathrm{cm}^{-3}$ (which is low compared to air with $10^{19}\,\mathrm{cm}^{-3}$). In figure 2 one can see the first experimental observation of BEC in 1995 at JILA in Boulder, Colorado. At first the BEC is located inside a trap. In the next step the trap is turned off and the atoms can follow their own momentum. Now one sends a light pulse on the gas of particles. Behind the particles there is a detector. So when all the particles remain at the same place one can see a dark spot on the detector. This means that the major part of particles had no momentum which means BEC has occurred.

We have briefly described the state of matter BEC. In the next chapter we focus on the case that the bosons interact with each other. This is described by the Gross-Pitaevskii-Equation.

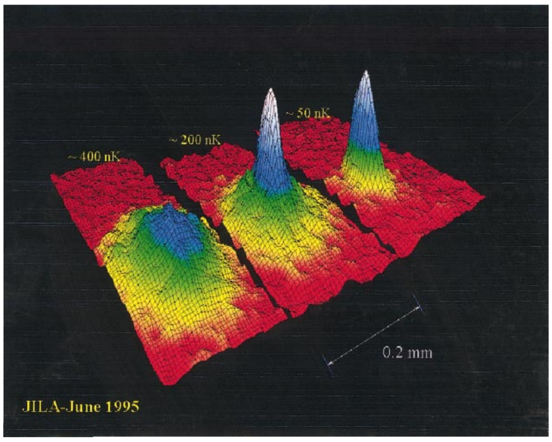

Figure 2: Typical image showing the observation of a Bose-Einstein-Condensate. Shown is the momentum distribution of the particles. Left: before the gas condenses; middle: partly condensed gas and right: almost all atoms are part of the condensate and have no momentum(Image from the first observation at JILA [CW02])

2 Gross-Pitaevskii-Equation

In this chapter I want to describe the Gross-Pitaevskii-Equation. A full derivation can be found in [ESY10]. The GPE is a nonlinear Schrödinger equation which describes BEC with atomic interactions. Because it is nonlinear it is hard to solve. For the one-dimensional case there exist analytical solutions called solitons which will be discussed later.

The time-independent GPE is given by

$$\mu\Psi(\mathbf{r}) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g|\Psi(\mathbf{r})|^2\right)\Psi(\mathbf{r}), \tag{5}$$

with μ the chemical potential, $\hbar$ the reduced Planck constant, ∇ the nabla operator, m the particle mass and $V_{ext}(\mathbf{r})$ the external trapping potential. The term $g|\Psi(\mathbf{r})|^2$ describes the atomic interaction; the so called coupling factor g is given by

$$g = \int V_{eff}(\mathbf{r}, t)d\mathbf{r} = \frac{4\pi\hbar^2 a}{m} \tag{6}$$

with the scattering length a. In equation (5) the eigenvalue of the wave function is the chemical potential μ and not the energy as for the normal Schrödinger equation (SE). For $g = 0$ equation 5 becomes the classical SE with a different eigenvalue. The wave functions is normed by

$$\int_{-\infty}^{\infty} |\Psi(\mathbf{r})|^2 dx^2 = 1. \tag{7}$$

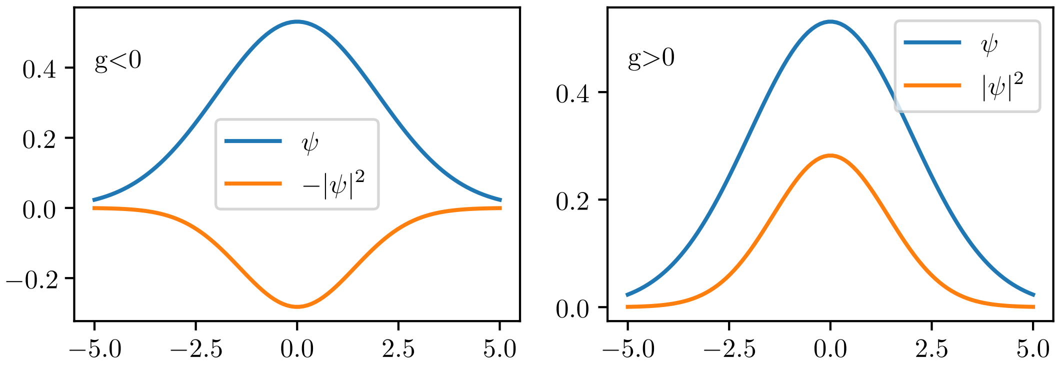

Figure 3: A Gaussian wave packet $\Psi(\mathbf{x}) = \frac{1}{\pi^{1/4}\sqrt{\sigma}} \exp(\frac{-x^2}{2\sigma^2})$ with the width $\sigma = 2$. In the left figure a potential $V_{eff} = -|\Psi(\mathbf{x})|^2$ is illustrated; in the right one with positive sign.

Substituting μ with the derivation of time $i\hbar\frac{\partial}{\partial t}$ we obtain the time-dependent GPE

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r},t) + g|\Psi(\mathbf{r},t)|^2\right)\Psi(\mathbf{r},t). \tag{8}$$

Again we have a kinetic term, an external potential and the term $g|\Psi(\mathbf{r},t)|^2$. The meaning of the first two is clear and in the following we discuss the case where the external potential is $V_{ext} = 0$. The latter one is also a potential and described in the following.

We know that the absolute square of the wave function is the probability density in position space. So the potential $g|\Psi(\mathbf{r},t)|^2$ has the form of the probability density in position space. It is a function of position $\mathbf{r}$ and time t.

In figure 3 one can see a Gaussian wave packet and the probability density $|\Psi(\mathbf{r},t)|^2$ times a factor $g = -1$ and $g = +1$.

Now consider a wave packet moving to the right. The probability density is moving to the right as well. So $g|\Psi(\mathbf{r},t)|^2$ describes a potential that is coupled to the wave function. It also changes the width when the wave function changes the width. For $g > 0$ the GPE describes repulsive interaction and for $g < 0$ attractive repulsion. This is obvious when looking at figure 3.

For a wave function $\Psi(\mathbf{r},t)$ which is an eigenstate of the interaction potential $g|\Psi(\mathbf{r},t)|^2$ the configuration is stable and should not change in time. This is the case for a special solution of the GPE called Solitons.

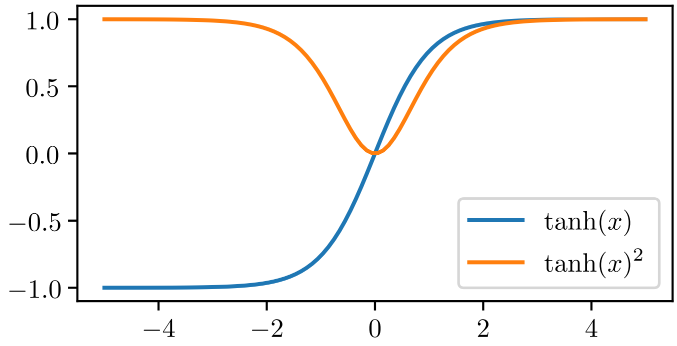

Figure 4: The dark soliton solution and its density distribution.

2.1 Solitons

[PS08] A soliton/solitary wave is a wave propagating in a nonlinear medium without changing its form. Today there are many theoretical and experimental studies about solitary waves. The first description of solitons dates back to 1834. The engineer John Scott Russell was riding his horse next to a canal where he observed a wave that almost did not change its form. This phenomenon of solitons in water is described mathematically by the Kortewegâde Vries equation which dates back to 1895. In 1973 the existence of solitons in optical fibre was predicted and experimentally shown in 1980. [Wik18]

Solitons are solutions for different nonlinear wave equations and the GPE is one of them. There are several kinds of solitons. In the following we differ bright solitons from dark solitons. Bright soliton have a positive amplitude; an example would be a light pulse in a fibre. Dark solitons are solitary waves with a negative amplitude, for example in matter.

The dark soliton solution of the stationary 1D GPE has the form

$$\Psi(x) = \Psi_0 \tanh\left(\frac{x}{\sqrt{2}\xi}\right) \tag{9}$$

with the coherence length

$$\xi = \frac{\hbar}{(2mn_0 g)} \tag{10}$$

where n_0 is the density of the condensate.

This wave function is shown in figure 4. For the time-dependent GPE (8) one can insert the ansatz

$$\Psi(x,t) = f(x-ut)\exp(\frac{-i\mu t}{\hbar}) \tag{11}$$

with velocity u and obtain the differential equation

$$-\frac{\hbar^2}{2m} f'' + g|f|^2 f = -i\hbar u f' + \mu f, \tag{12}$$

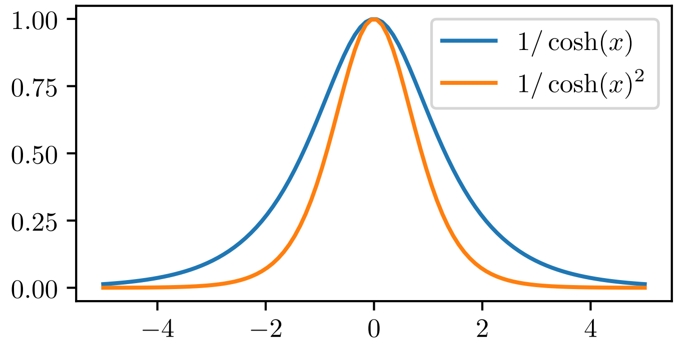

Figure 5: The bright soliton solution and its density distribution.

where f' is the derivation $\frac{df}{d(x-ut)}$. Solving this equation leads to the soliton wave function [PS08]

$$\Psi(x,t) = \sqrt{n_0}\left[i\frac{u}{s} + \sqrt{(1-\frac{u^2}{s^2})}\tanh\left(\frac{x-ut}{\sqrt{2}\xi_u}\right)\right]\exp(\frac{-i\mu t}{\hbar}), \tag{13}$$

where $s = (n_0 g/m)^{1/2}$ is the sound velocity of the condensate and

$$\xi_u = \frac{\xi}{[1-(u/s)^2]^{1/2}} \tag{14}$$

describes the width. The density distribution is given by

$$|\Psi(x,t)|^2 = n_0\left[i\frac{u^2}{s^2} + (1-\frac{u^2}{s^2})\tanh^2\left(\frac{x-ut}{\sqrt{2}\xi_u}\right)\right]. \tag{15}$$

The bright soliton solution has the form

$$f = \frac{b}{\cosh(ax)} = b\,\mathrm{sech}(ax) \tag{16}$$

where a and b are given by

$$a^2 = -\frac{2m\mu}{\hbar^2} \tag{17}$$

and

$$b^2 = \frac{2\mu}{g}. \tag{18}$$

It is shown in figure 5. A requirement for this solution is that $\mu < 0$ and $g < 0$. So the wave function of the bright soliton at rest is

$$\Psi(x,t) = \left(\frac{2\mu}{g}\right)^{1/2}\frac{1}{\cosh[(2m|\mu|/\hbar^2)^{1/2}x]}\exp(\frac{-i\mu t}{\hbar}) \tag{19}$$

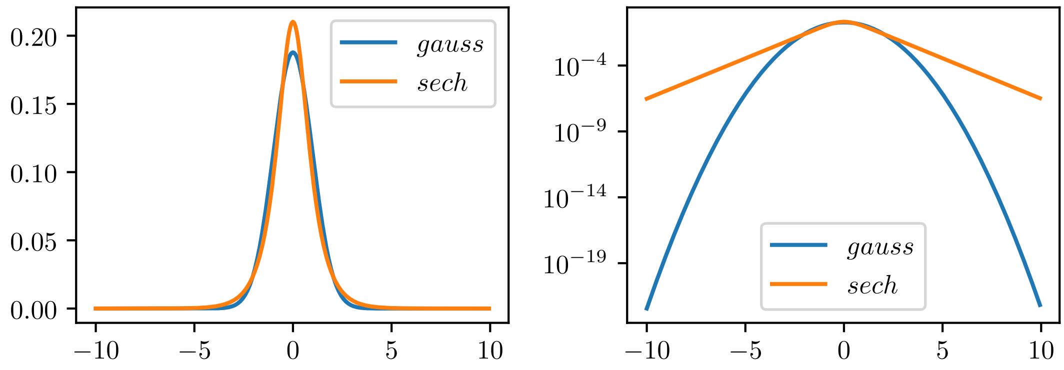

Figure 6: Comparison of the exact bright soliton solution (sech) to the approximation by a Gaussian distribution (gauss) on linear and logarithmic scaling.

with its density distribution

$$|\Psi(x,t)|^2 = \left(\frac{2\mu}{g}\right) \frac{1}{\cosh^2[(2m|\mu|/\hbar^2)^{1/2}x]}. \tag{20}$$

2.2 Model of a soliton

This thesis investigates bright solitons modelled by a Gaussian wave packet. The wave function of the approximated soliton (AS) is given by

$$\Psi(x) = \frac{1}{\pi^{1/4}\sqrt{\sigma}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \tag{21}$$

where the density distribution is given by

$$|\Psi(x)|^2 = \frac{1}{\pi^{1/2}\sigma} \exp\left(\frac{-x^2}{\sigma^2}\right). \tag{22}$$

In figure 6 one can see the difference between the exact soliton solution (19) and the approximation (21). Major differences of the two are the maximum amplitude and the behaviour close to zero.
The potential $g|\Psi(\mathbf{r},t)|^2$ for a Gaussian wave packet can be approximated using Taylor expansion:

$$g|\Psi(\mathbf{r},t)|^2 \approx \frac{g}{\sqrt{\pi}\sigma} - \frac{gx^2}{\sqrt{\pi}\sigma^3} + \mathcal{O}(x)^3. \tag{23}$$

We approximate the potential to be harmonic as $g|\Psi(x)|^2 \approx \frac{gx^2}{\sqrt{\pi}\sigma^3}$ so we can easily estimate the ground state.

We have the quantum harmonic oscillator with the Schrödinger equation:

$$\mu\Psi(x) = \left(-\frac{\hbar^2\nabla^2}{2m} + \frac{gx^2}{\sqrt{\pi}\sigma^3}\right)\Psi(\mathbf{r}), \tag{24}$$

for which we obtain the ground state

$$\Psi_0(x) = \frac{1}{\pi^{1/4}\sqrt{\sigma}}\exp\left(\frac{-x^2}{2\sigma^2}\right) \tag{25}$$

with

$$\sigma = \sqrt{\frac{\hbar^2}{2m\mu}}. \tag{26}$$

Consequential we obtain the requirement for the coupling factor g

$$g = \frac{\hbar^2\sqrt{\pi}}{2m\sigma}. \tag{27}$$

Using these parameters the Gaussian wave packet should be an eigenstate of the potential and so the wave should be stable. Testing it in the simulation did not bring the expected result. Reasons might be the approximation of the potential by a Taylor series or the normalization of the coupling factor. So I checked if there is a configuration with stable Gaussian waves by heuristic means. In one dimension it is possible to find one:
For good parameters we get the results that are shown in figure 7 for the Gaussian wave packet and figure 8 for the exact result. As one can see the Gaussian approximation has some little disturbances but it is still stable. The 2D case will be discussed later in this thesis.

2.3 Units

For the simulation we need to choose reasonable units for the time t and the coupling factor g. This is derived here. In the following $\bar{x}$ and $\bar{m}$ are the characteristic length and mass scale.
The absolute square of the wave function $|\Psi(\mathbf{r},t)|^2$ is the probability density and has the unit $1/\bar{x}^{dimension}$. Because we are in two dimensions we find

$$|\Psi(\mathbf{r},t)|^2 = \frac{1}{\bar{x}^2}. \tag{28}$$

The term $g|\Psi(\mathbf{r},t)|^2$ in equation (8) behaves like a potential and so it has the unit of energy:

$$g|\Psi(\mathbf{r},t)|^2 = \frac{\bar{m}\bar{x}^2}{t^2}. \tag{29}$$

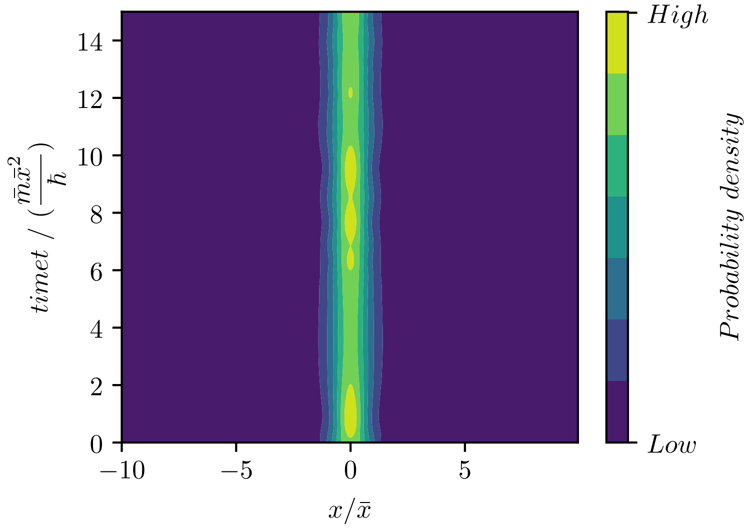

Figure 7: A soliton approximated by (21). There are some little disturbances but the soliton remains stable.

Using equation (28) and (29) we obtain the unit of g:

$$g = \frac{\bar{m}\bar{x}^4}{t^2}. \tag{30}$$

The kinetic part of the Gross-Pitaevskii-Equation (8) $\frac{\hbar^2\nabla^2}{2m}$ has also the unit of energy:

$$\frac{\hbar^2\nabla^2}{2m} = \frac{\bar{m}\bar{x}^2}{t^2}. \tag{31}$$

We substitute ∇^2 with $\bar{x}^{-2}$ and we can write the unit of the time as

$$t = \frac{\bar{m}\bar{x}^2}{\hbar}. \tag{32}$$

3 QuantumOptics.jl

So far we discussed the properties of BEC, GPE and solitons. Now we investigate the question if there is a stable soliton solution in the 2D GPE. We simulate the problem by numerical methods in the framework QuantumOptics.jl (QO.jl). QO.jl runs on the programming language Julia. Both are presented shortly.

3.1 Julia

"Julia is a high-level, high-performance dynamic programming language [...]"[BEKS17] which aims to be simple but still efficient. It is designed for numerical analysis

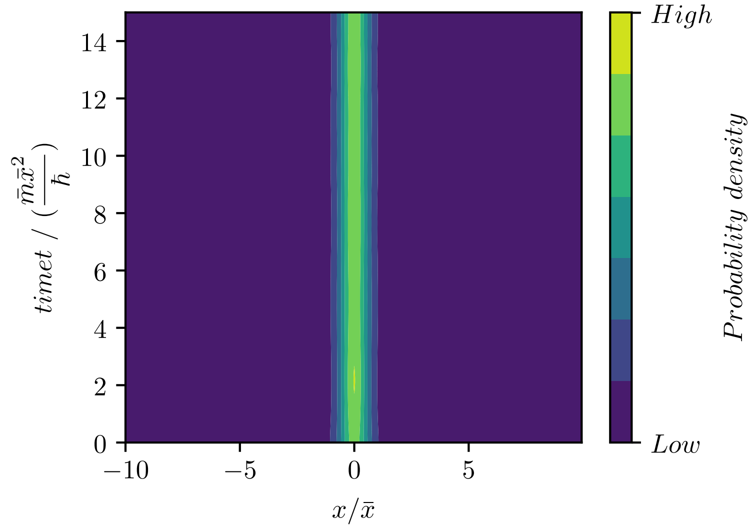

Figure 8: The analytical result (19) for the Gross-Pitaevskii Equation remains stable over time.

and computational science. The programming language comes with many packages involving the subjects physics, mathematics and analysis, geology, economy, chemistry, robotics and many more.
Some interesting packages for physicists are

- **QuantumOptics.jl** for simulation of closed and open quantum systems. It will be discussed in detail.
- **DifferentialEquations.jl** which can be used for the numerical solution of differential equations.
- **Measurements.jl** for errors computation and calculation of variables with uncertainties.
- **PhysConst.jl** provides the most important constants together with uncertainties, units etc.
- **Unitful.jl** for the calculation with units.
- **PyCall.jl** for directly calling and interoperating with Python. You can use Python functions, import Python modules and share data structures between Python and Julia.

On https://juliaobserver.com/ there is a ranking of the most popular packages sorted by subject. One last thing to mention is the possibility to run Julia in the browser-based application Jupyter.

3.2 QuantumOptics.jl

[KPOR17] QuantumOptics.jl is an open source computational framework built for the simulation and investigation of open quantum systems. It is built in Julia and aims to be intuitive to use with better performance than other alternatives like Quantum Toolbox in Python (QuTip) or Quantum Optics Toolbox in Matlab (QO). The architecture is modelled on QuTip but has some fundamental differences.

In QuantumOptics.jl there are different bases which are used for defining a problem. After defining a basis one uses quantum objects like states and operators which “know” in which base they are defined. It is not possible to apply an operator to a state that is not in the same basis by mistake. Furthermore you can apply various time evolutions or other operations on the quantum objects. It has to be mentioned that QO.jl comes with a comprehensive and very helpful documentation with many examples.
In the following I give a short introduction to this toolbox by presenting the code used for implementing two dimensional solitons.

4 Code

In this thesis the version of Julia is 0.6.3 and QuantumOptics.jl is used in version 0.5.2. For the Numerical Simulation of a given problem in QuantumOptics.jl one can follow the steps as described below.
At first the QuantumOptics.jl framework has to be imported into Julia. This is done by

```
using QuantumOptics, PyPlot, PyCall, JLD
```

where PyPlot and PyCall are used for plotting the data. The package JLD can be used to save data into a separate file.

4.1 Defining a Basis

Now we go on by choosing a basis which represents the dimensions of our Hilbert-Space. For the implementation of various systems there are the following predefined bases:

- **Particle** for a system in position and momentum space.
- **Spin** for a spin-system of arbitrary spin number.
- **Fock** for implementing a fock space with a variable number of particles.

- **N-Level** for an atom which can be reduced to a few relevant levels.
- **Many-Body** for describing a system of many identical particles.
- **Subspace** can be used to restrict a large Hilbert-Space to a subspace including all essential information.
- **GenericBasis** can be used when your system is not defined yet. It just needs to know the dimension of your Hilbert-Space. Another option is to define a new basis type.

We use the **Particle**-Basis for defining position and momentum space in x:

```
x_min= -10
x_max = 10
x_steps = 32
dx = (x_max - x_min) / x_steps
b_x = PositionBasis(x_min, x_max, x_steps)
b_px = MomentumBasis(b_x)
xsample = samplepoints(b_x)
```

with a resolution of 32 in x direction. It is reasonable to use powers of 2 for better performance.
Because we want the simulation in 2D we also need a position and momentum space in y-direction

```
y_min= -10
y_max = 10
y_steps = 32
dy = (y_max - y_min) / y_steps
b_y = PositionBasis(y_min, y_max, y_steps)
b_py = MomentumBasis(b_y)
ysample = samplepoints(b_y)
```

where we used the same size and steps as in x.
At this point these two bases are not yet composite. One can create a composite basis with the `tensor()`-function or the equivalent unicode symbol $\otimes$ as follows.

```
b_comp_x = tensor(b_x, b_y)
b_comp_p = b_px ⊗ b_py
```

4.2 Operators

The size of the Hilbert-Space becomes large in two dimensions and computing time can be long. For better performance it is convenient to use a Fourier-Transformation for switching from Position to Momentum Space. This is done by

```
Txp = transform(b_comp_x, b_comp_p)
Tpx = transform(b_comp_p, b_comp_x)
```

Our Hamiltonian

$$H = \frac{\hbar^2\nabla^2}{2m} + g|\Psi(\mathbf{r},t)|^2 = \frac{\mathbf{p}^2}{2m} + g|\Psi(\mathbf{r},t)|^2 \tag{33}$$

can be defined as follows. First we need the momentum $\mathbf{p}$

```
px = momentum(b_px)
py = momentum(b_py)
```

in x and y.
Next we define the kinetic term $\frac{\mathbf{p}^2}{2m}$ of the Hamiltonian in x and in y:

```
Hkinx = LazyTensor(b_comp_p, [1, 2], [px^2/2, one(b_py)])
Hkiny = LazyTensor(b_comp_p, [1, 2], [one(b_px), py^2/2])
Hkinx_FFT = LazyProduct(Txp, Hkinx, Tpx)
Hkiny_FFT = LazyProduct(Txp, Hkiny, Tpx)
```

with the more efficient Fourier Transformation. We have set $m = 1$. The `Lazy`-Operators "allow delayed evaluation of certain operations. This is useful when combining two operators is numerically expensive but separate multiplication with states is relatively cheap." [doc]
The interaction potential $|\Psi(\mathbf{r},t)|^2$ is defined by

```
HΨ =diagonaloperator(b_comp_x, Ket(b_comp_x).data)
```

and the full Hamiltonian is the sum of all parts.

```
H0 = LazySum(Hkinx_FFT, Hkiny_FFT, HΨ)
```

4.3 State

Now we go on by defining our initial state. As already mentioned we model a soliton with a Gaussian wave packet. For now we want a state without momentum so

```
x0 = 0
y0 = 0
p_x = 0
```

```
p_y = 0
σ =1.5
```

are our constants. `x0` and `y0` describe the initial position; `p_x` and `p_y` describe the initial momentum and σ is the width of the wave packet. For the wave packet itself QuantumOptics.jl has a pre-defined function so it is easy to create one

```
Ψx =gaussianstate(b_x, x0, p_x, σ)
Ψy =gaussianstate(b_y, y0, p_y, σ)
Ψ =normalize(Ψx ⊗ Ψy)
```

where Ψ is our wave packet in 2 dimensions.

4.4 Time Evolution

In QO.jl there are different time evolutions that are based on DifferentialEquations.jl:

- `timeevolution.schroedinger()` for solving the classical Schrödinger equation describing closed quantum systems. There is the possibility of using a time-dependent Hamiltonian with `timeevolution.schroedinger_dynamic()`.
- `timeevolution.master()` for open quantum systems described by the master equation in Lindblad form
- `timeevolution.mcwf()` is another option for open quantum systems. It uses the Monte Carlo wave function (MCWF) method. "For large numbers of trajectories the statistical average then approximates the result of the Master equation."

Our Hamiltonian is time-dependent and so we need to create a function for updating the term $g|\Psi(\mathbf{r},t)|^2$:

```
function H(t, Ψ)
    HΨ.data.nzval .=g*abs2.(Ψ.data)
    return H0
end
```

After defining our coupling factor g and the time span T

```
T = [0:1:15;]
g = 0
```

we use `timeevolution.schroedinger_dynamic()` for time evolution.

```
tout, Ψt =timeevolution.schroedinger_dynamic(T, Ψ, H)
density = [reshape(abs2.(Ψ.data), (x_steps, y_steps))' for Ψ=Ψt]
```

and the **reshape()**-function for creating a matrix for plotting. We obtain the probability density of the wave function at different times.

5 Results

This chapter discusses the results of the simulation using QuantumOptics.jl. We want to find out whether stable solitons approximated by a Gaussian wave packet exist in 2D. At first the behaviour of one soliton is investigated. Then we discuss the dynamics of two coupled solitons. The chapter ends with a method by [SM06] for stabilizing solitons in 2D. All figures come with a video showing more frames of the simulation. One thing to note is that the time steps in the videos differ.

Two dimensional solitons approximated by a Gaussian wave in the GPE are found to be not stable. First we discuss the meaning of the coupling factor. For a resolution of 32 steps in x, 32 steps in y and a width $\sigma = 2$ the dependence of the coupling factor g is as follows. For too small g the soliton expands and for too strong g a contraction and a problem in the simulation occurs: the norm of the wave function starts to grow. From zero up to a coupling factor of $g = -15.1\,[\bar{m}\bar{x}^4/t^2]$ the time of stability of the soliton increases. Further increasing the factor g still increases the time of stability but also brings the effect of the contraction starting at $g = -15.2\,[\bar{m}\bar{x}^4/t^2]$ after about 120 time units. From there increasing g quickens the wave to contract. This dependence of the time of stability to the factor g is shown in Figure 9.
In Figure 10 one can see a single soliton underlying the Gross-Pitaevskii Equation at four different times. The coupling factor g is zero. As expected the wave expands. Comparing this to a coupling factor of $g = -15.1\,[\bar{m}\bar{x}^4/t^2]$ in Figure 11 one can see a difference in time. For $g = 0$ the wave diverges after 6 time steps where for $g = -15.1\,[\bar{m}\bar{x}^4/t^2]$ the wave exists for 80 units of time. But it still expands after 80 time steps. Oscillation of the width of the wave can be seen when looking at more time steps.
In Figure 12 one can see the behaviour for $g = -15.2\,[\bar{m}\bar{x}^4/t^2]$. The soliton looks good at 80 time units. But after 119 time steps the soliton “contracts“ and the norm of the wave function starts to grow.

Increasing the resolution of the simulation brings an earlier expansion of the wave and decreasing brings the contraction of the wave. The smaller one chooses the resolution the earlier occurs the contraction in this configuration.

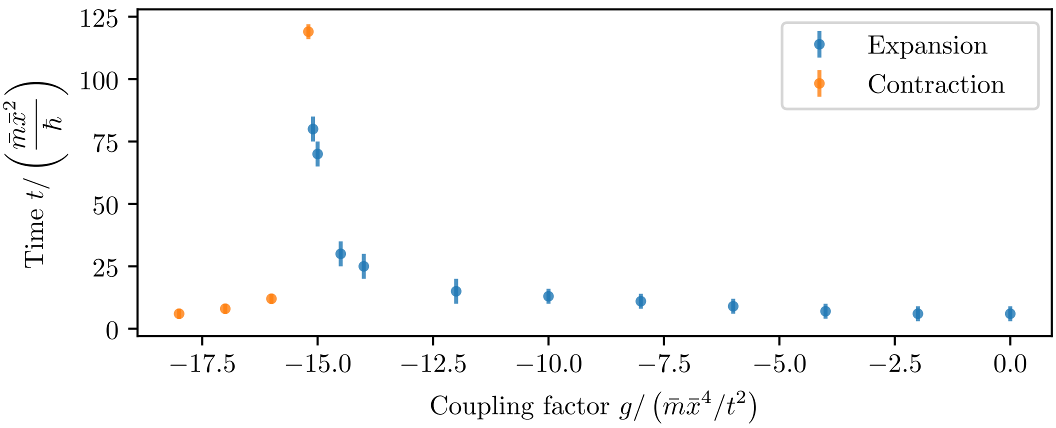

Figure 9: The stability time of the soliton in dependence of the coupling factor g. For a coupling factor up to -15.1 the wave expands and from there a contraction occurs.

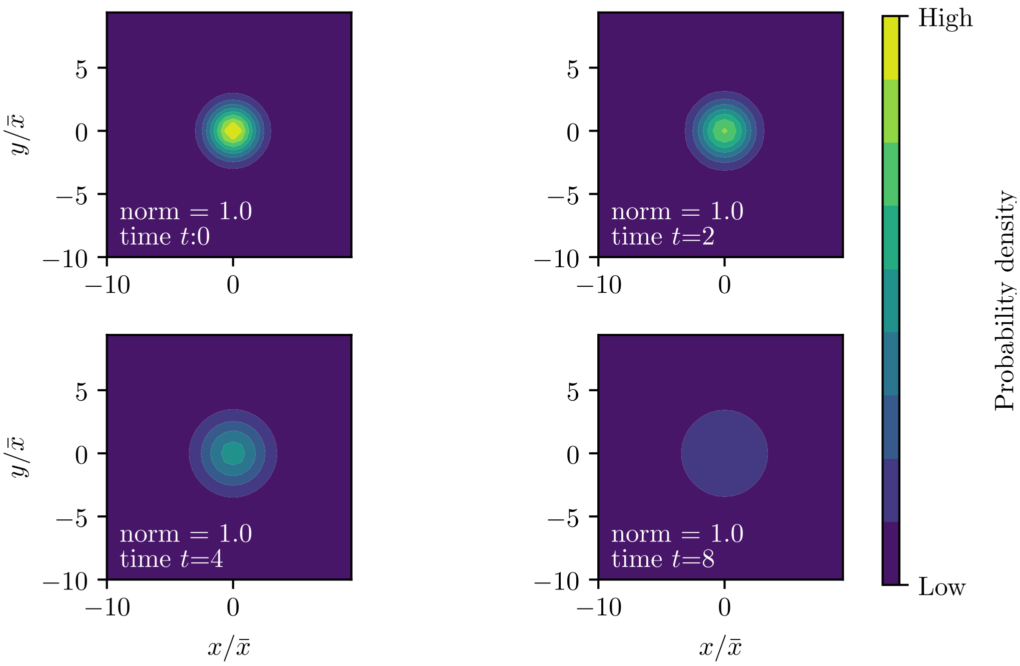

Figure 10: A single 2D Gaussian wave in the Gross-Pitaevskii Equation. The coupling factor is chosen to be zero. One can clearly see how it expands. [video]

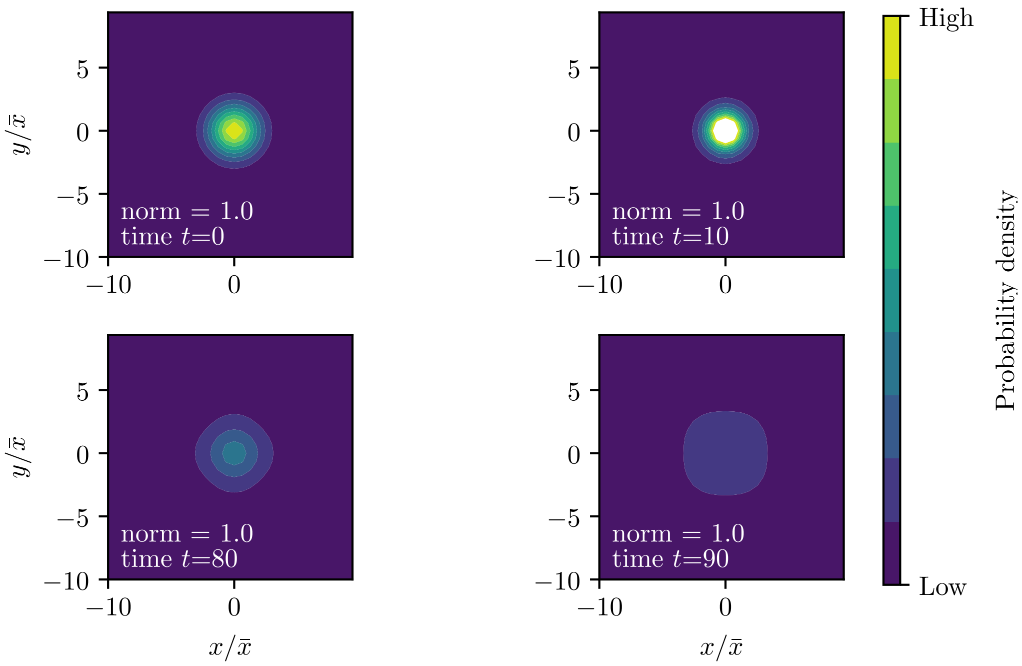

Figure 11: Here the coupling factor is $g = -15.1$. It takes more time to expand than for g being zero. [video]

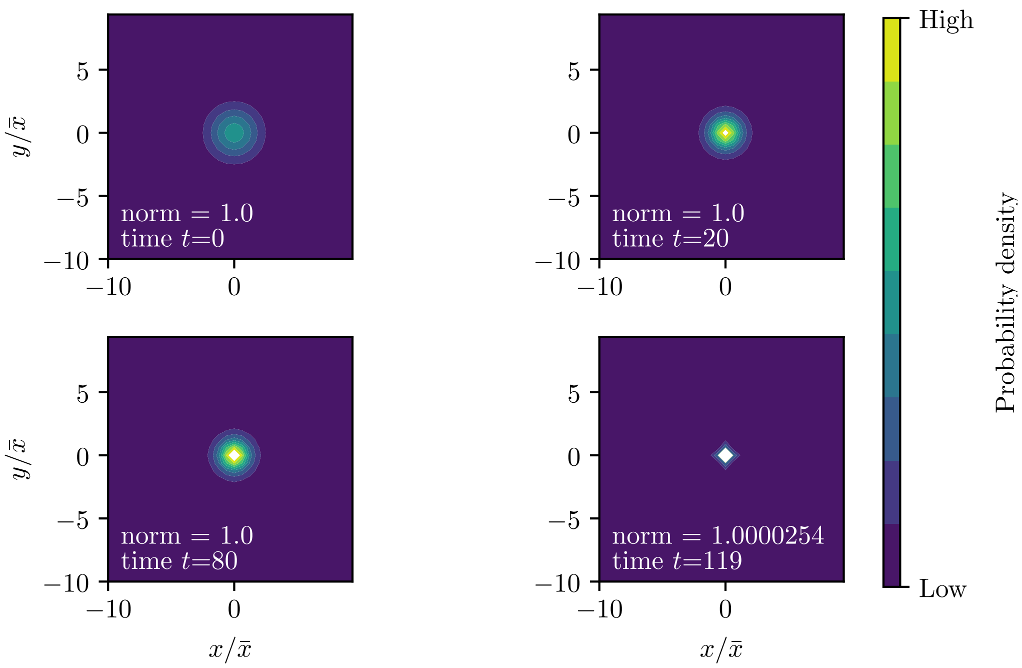

Figure 12: For $g = -15.2$ the wave contracts after 119 time steps and problems in the simulation occur. One can see the contraction and that the norm starts to grow in the last plot. [video]

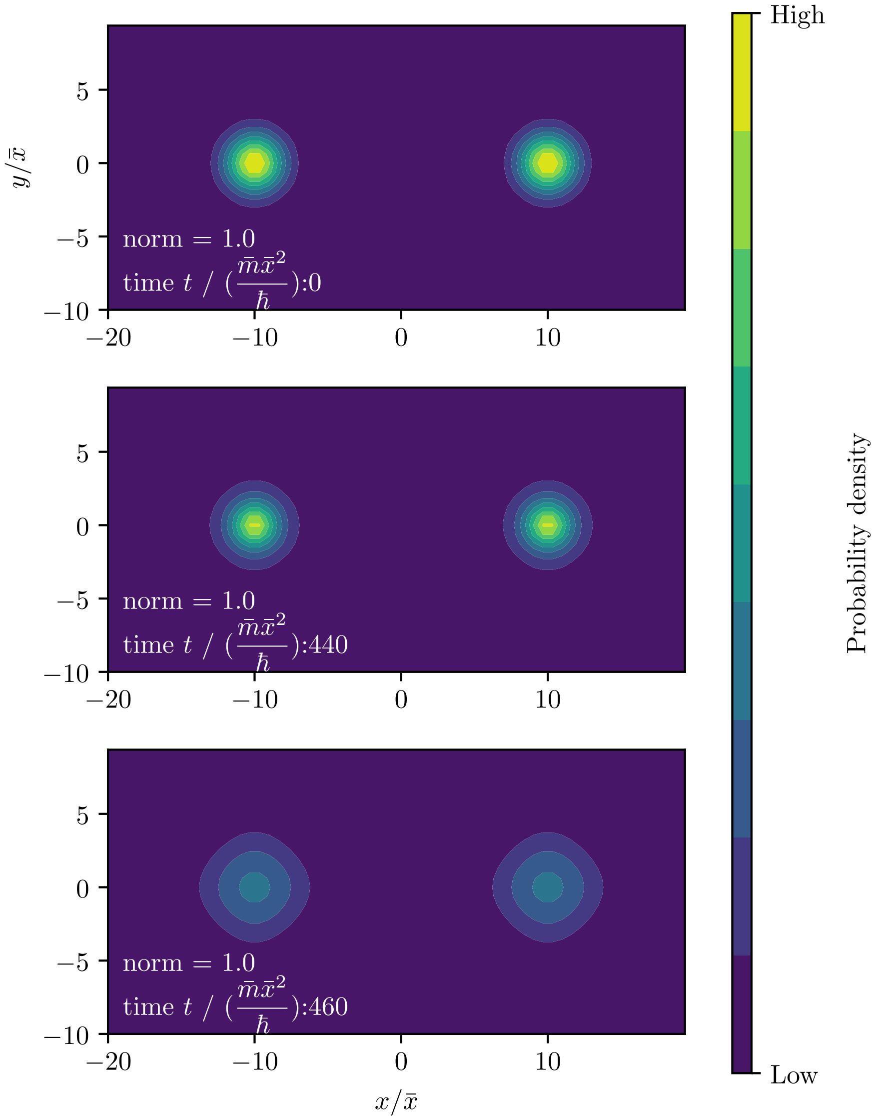

Figure 13: Two coupled solitons remain stable for a longer time than a single one. ($g = -29.45$) [video]

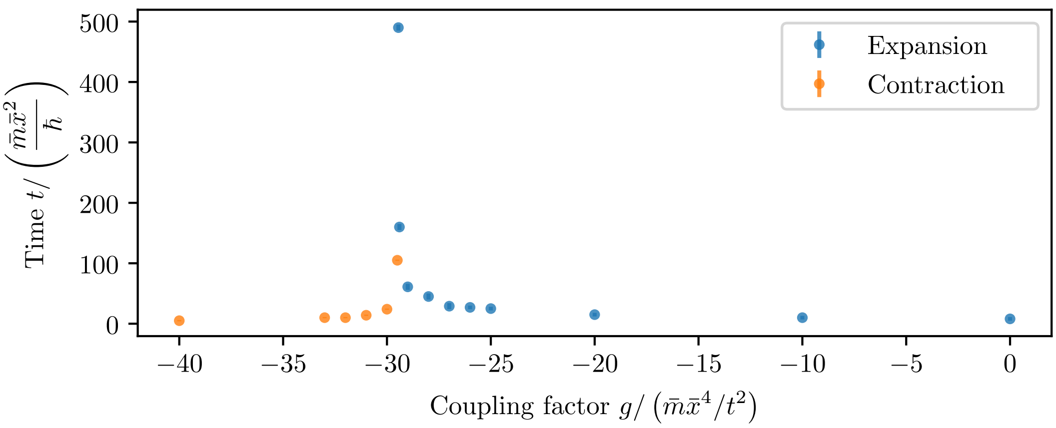

Figure 14: The stability time of two coupled solitons in dependence of the coupling factor g. For a coupling factor up to -29.45 the wave expands and from there a contraction occurs.

5.1 Two solitons at rest

Now we investigate what happens when we create two solitons that are coupled to each other. The resolution is now 62 steps in x and 32 steps in y direction. In Figure 13 one can see two coupled solitons which are a distance of $x = 20$ apart. This configuration is stable for a longer time than a single soliton. It is stable up to 460 time steps but then starts to expand. The same configuration with the two solitons closer together is shown in this video . Here the attraction is too strong and the solitons merge to a single one. This single soliton contracts because for a single soliton the coupling factor g is too strong.
The relation between the time of stability for two solitons and the strength of g is shown in Figure 14. It shows similar behaviour as for a single soliton.

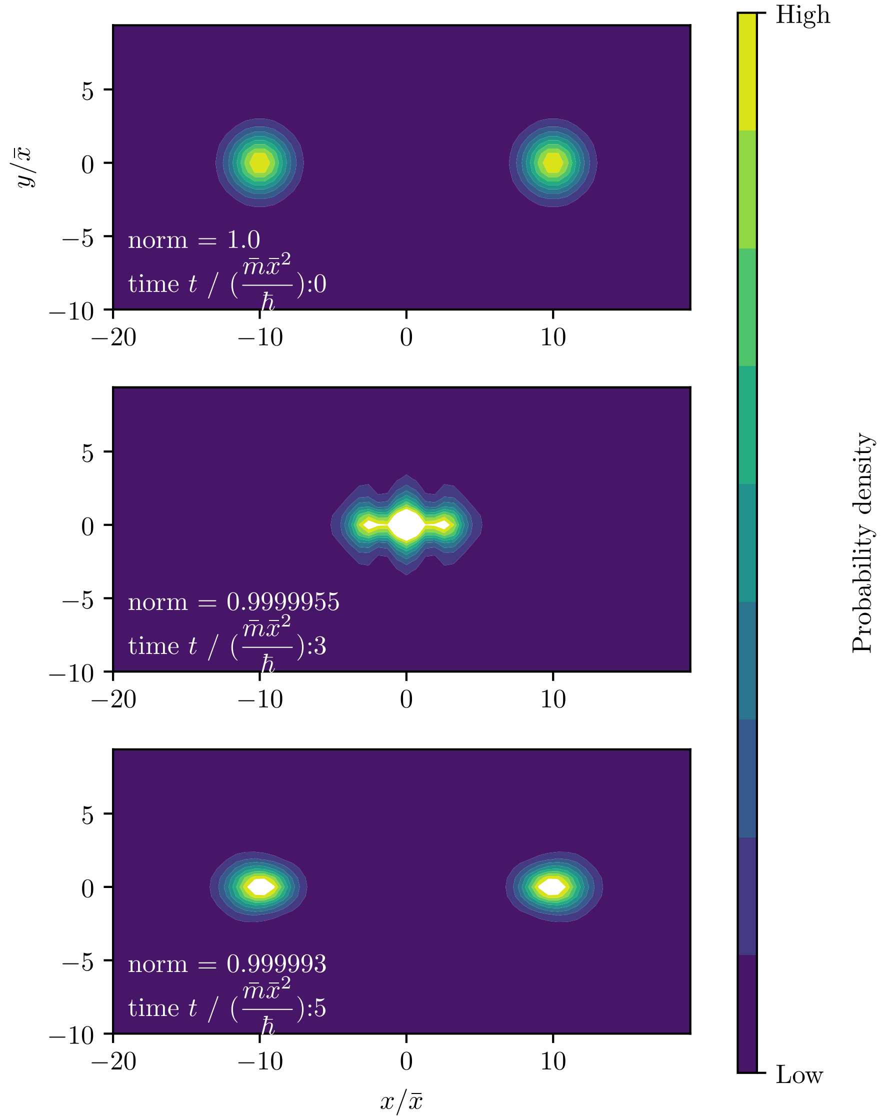

Figure 15: Two solitons with opposite momentum. The coupling factor is $g = -29.45$ and the momentum is $p = 4$. [video]

5.2 Collision of two solitons

Here we discuss the collision of two solitons. We have a set of two coupled solitons that both have the same momentum but in opposite directions. Having a momentum of $p = 4\,\bar{m}\bar{x}/t$ brings the result shown in Figure 15. When colliding the two solitons show interference and then separate. After separating they do not expand but problems with the norm occur. For a momentum of $p = 1\,\bar{m}\bar{x}/t$ the behaviour is shown in this video. When colliding they again show interference but then stretch and expand at the same time.

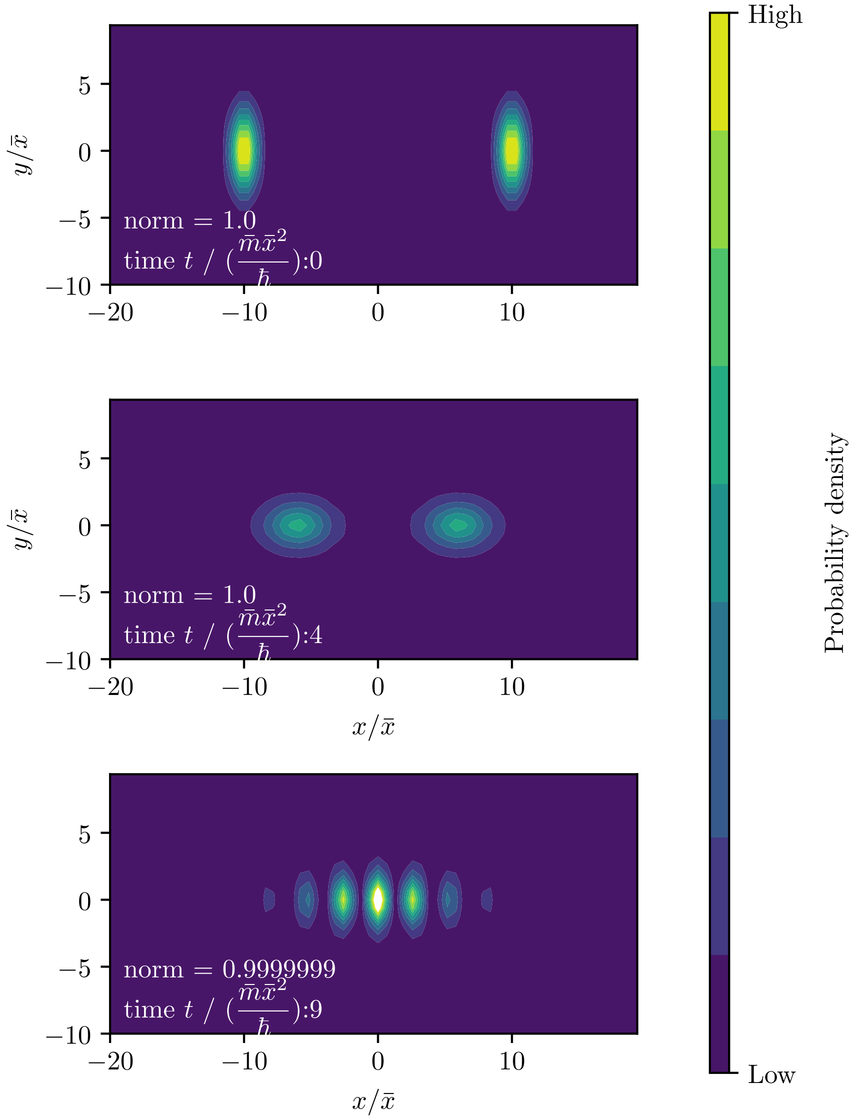

Figure 16: Two solitons that have the shape of an ellipse. In the beginning σ_y is 3 times larger than σ_x. We have $g = -29.45$ and $p = 1$. [video]

5.3 Other configurations

Now we change the shape of the initial state to an ellipse. We have a Gaussian wave packet with $\sigma_x = 1$ and $\sigma_y = 3$. Without momentum the two solitons become round and expand after 10 time steps. In Figure 16 two such waves have opposite momentum $p = 1$. The solitons become round and when contracting they show a stronger interference than for the configuration in Figure 15. After this they expand.

Another configuration is two solitons brushing each other. The two solitons collide

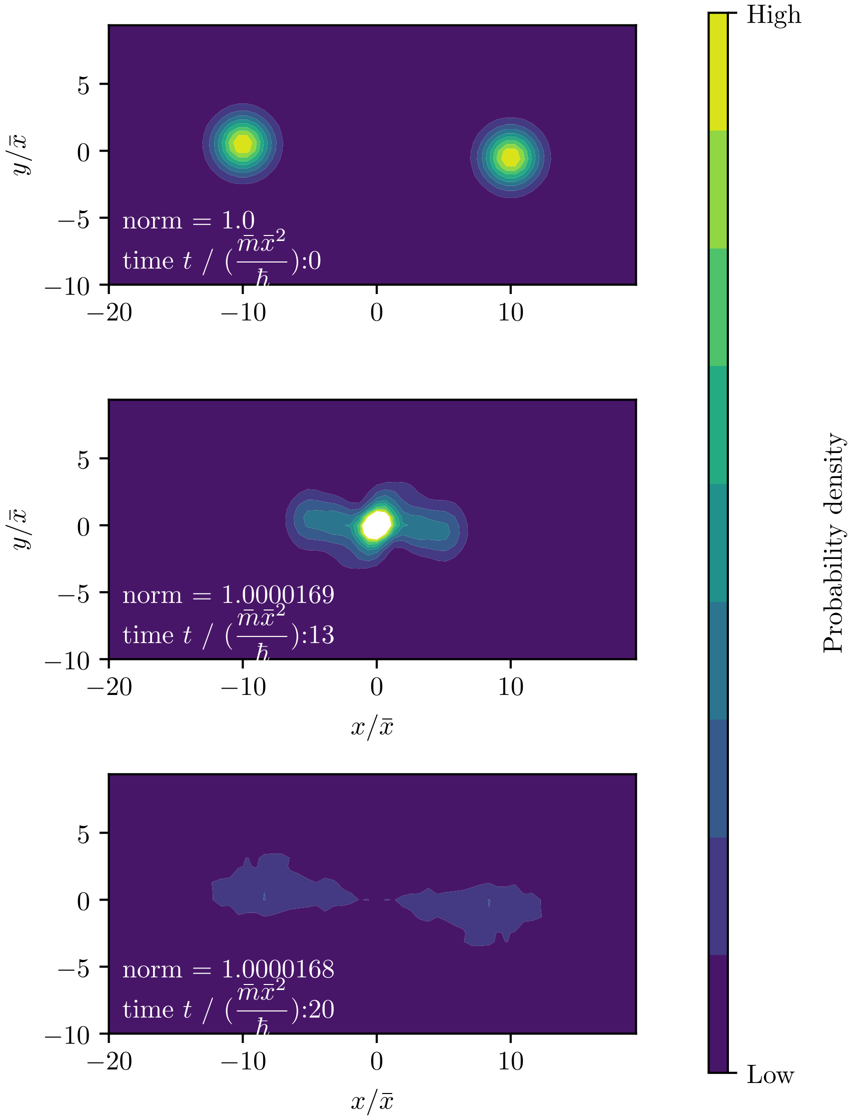

Figure 17: Two solitons brushing each other. the coupling factor is $g = -29.45$ and the momentum is $p = 1$. [video]

and show interference but afterwards they expand. One thing to mention is that they have swapped their momentum. The upper one now has a momentum to the left.

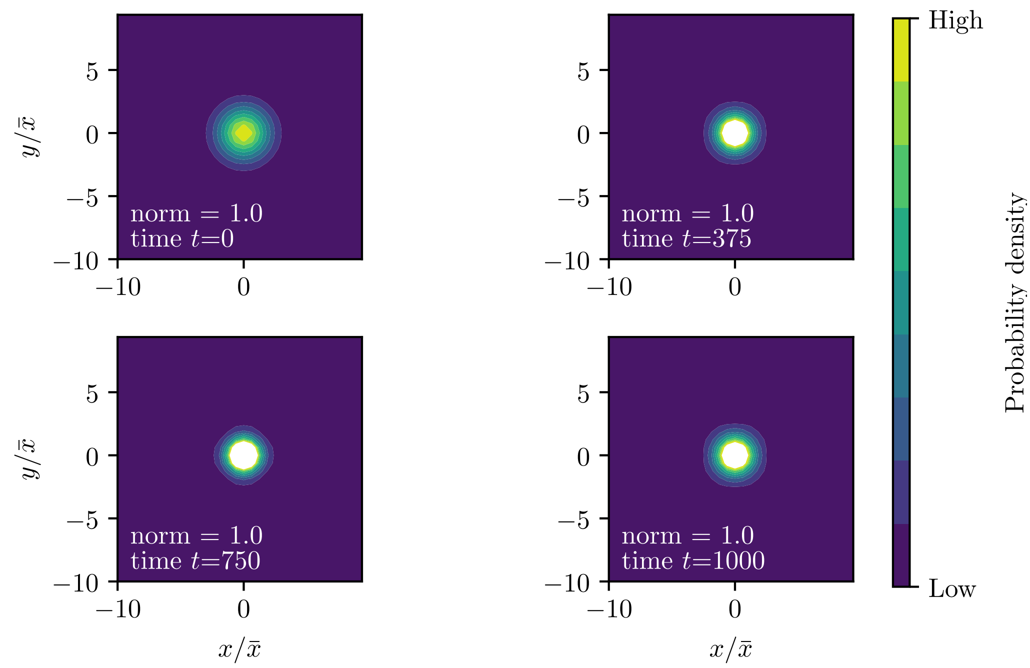

Figure 18: The nonlinearity is restricted to a region with radius $r < 2$. The soliton remains stable for the full time tested. [video]

5.4 Method by Sakaguchi and Malomed

Hidetsugu Sakaguchi and Boris Malomed published a paper [SM06] on how to stabilize 2D solitons in the Gross-Pitaevskii Equation. They show how to spatially restrict the nonlinearity of the GPE. The GPE then has the form

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r},t) + g(r)|\Psi(\mathbf{r},t)|^2\right)\Psi(\mathbf{r},t). \tag{34}$$

with the nonlinearity coefficient

$$g = \begin{cases} 1 & r < R \\ 0 & else \end{cases}$$

where ρ and R are parameters for restricting the region of the nonlinearity. In this thesis this ansatz to stabilize the solitons was tested with the parameter $R = 2$. The coefficient g was chosen to be $g = -15.1$ as this coupling strength worked well for a single soliton. In Figure 18 one can see the result at times $t = 0,\ 375,\ 750\ and\ 1000$. The soliton remains stable for the full time tested and the norm remains 1. Hence, our work confirms the method of [SM06] for stabilizing solitons.

6 Conclusion

We numerically studied solitons approximated by a Gaussian distribution in the QuantumOptics.jl toolbox. The implementation in the toolbox was very intuitive and straightforward. In one dimension the approximation of a Gaussian distribution brings stable solitons even though some disturbances occur.
In two dimensions the solitons are found to be unstable. For weak coupling factor g the wave function expands and for strong g contraction occurs. Interestingly two coupled solitons are stable for a longer time than a single soliton.
The collision of two solitons shows typical interference pattern. For small momentum the wave function expands during the collision but for stronger momentum the solitons separate afterwards.
The method by [SM06] for stabilizing solitons in two dimensions was confirmed for a single soliton.
A challenging open problem is the relation between the coupling factor g and the wave width σ. Another interesting aspect would be the comparison of the exact soliton solution $\propto \operatorname{sech}(x)$ to the Gaussian distribution in two dimensions. Additional features may arise by implementing other configurations such as e.g three or more coupled solitons or solitons with an angular momentum.

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